Chapter 6: The Wave Theory

PART 1

Waves on a water surface are such a familiar and expected occurrence that a completely still, glassy pool excites surprise and admiration (Fig. 6.1). You can also observe waves on flags being blown by a strong wind. In this chapter you will be concerned with how waves propagate, what properties are used to describe them, and how waves combine with one another when several pass through the same point in space at the same time. In the wave theory, which was formulated by Christian Huygens during the seventeenth century, the space and time distribution of waves is derived from two assumptions, the superposition principle and Huygens' Principle. The wave theory is very "economical" in the sense that far-reaching consequences follow from only these two assumptions.

Christian Huygens (1629-1695) was born at The Hague in Holland. His father Constantine, a man of wealth, position, and learning, quickly recognized the boy's unusual capabilities. Christian's father taught him both mathematics and mechanics, and long before his thirtieth birthday, Huygens had published important papers on mathematics, built and improved telescopes, discovered a satellite of Saturn, and invented the pendulum clock. In 1665, King Louis XIV of France invited Huygens to join the brilliant galaxy of intellects that the "Sun King" had clustered about him at Versailles. After 15 years in Paris, Huygens returned to spend his last years in Holland. These last years, however, proved to be as remarkable as his early years. In 1690, Huygens published the Treatise on Light*, his historic statement of the wave theory of light.*

Waves are important in physics because they have been used in the construction of very successful working models for radiation of all kinds. You can easily imagine that dropping a pebble into a pond and watching the ripples spread out to the bank suggests interaction-at-a-distance between the pebble and the bank. The waves are the intermediary in this interaction, just as radiation was the intermediary in some of the experiments described in Sections 3.4 and 3.5. In Chapter 7, we will describe wave models for sound and light and how these models can explain the phenomena surveyed in Chapter 5. The success of these models confirms Huygens' insight into the value of wave theory. However, Huygen's contributions and wave theory were not fully appreciated and exploited until the nineteenth century.

Waves were originally introduced as oscillatory disturbances of a material (called the *medium*) from its equilibrium state. Water waves and waves on a stretched string, the end of which is moved rapidly up and down, are examples of such disturbances. The waves are emitted by a source (the pebble thrown into the pond), they propagate through the medium, and they are absorbed by a receiver (the bank). Even though waves are visualized as disturbances in a medium, their use in certain theories nowadays has done away with the material medium. The waves in these applications are fluctuations of electric, magnetic, or

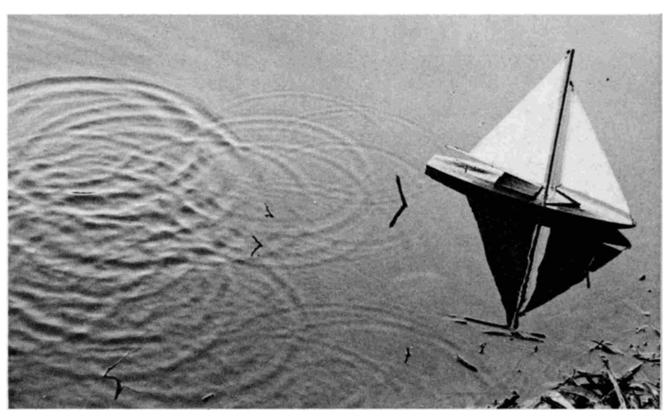


Figure 6.1 The reflected image gives information about the smoothness of the water surface. Why are the reflections of the sails dark and not white?

gravitational fields, rather than oscillations of a medium. The use of such waves to represent radiation has unified the radiation model and the field model for interaction-at-a-distance (Section 3.5). Our discussion here, however, will be of waves in a medium and not of waves in a field.

6.1 The description of wave trains and pulses

Oscillator model. We will analyze the motion of the medium through which a wave travels by making a working model in which the medium is composed of many interacting systems in a row. Each system is capable of moving back and forth like an oscillator, such as the inertial balance shown below and described in Section 3.4. You may think of the oscillators in a solid material as being the particles in an MIP model for the material.

Amplitude and frequency. Each oscillator making up the medium has an equilibrium position, which it occupies in the absence of a wave. When an oscillator is set into motion, it swings back and forth about the equilibrium position. The motion is described by an *amplitude* and a *frequency* (Fig. 6.2). The amplitude is the maximum distance of the oscillator from its equilibrium position. The frequency is the number of complete oscillations carried out by the oscillator in 1 second.

Interaction among oscillators. When waves propagate through the medium, oscillators are displaced from the equilibrium positions and are set in motion. The wave propagates because the oscillators interact with one another, so that the displacement of one influences the motion of the neighboring ones, and so on. Each oscillator moves with a frequency and an amplitude. It is therefore customary in this model to identify the frequency and amplitude of the oscillators with the frequency and amplitude of the wave. In addition, as you will see, there are properties of the wave that are not possessed by a single oscillator but that are associated with the whole pattern of displacements of the oscillators.

Conditions for wave motion. The oscillator model described above has two general properties that enable waves to propagate. One is that the individual oscillator systems interact with one another, so that a displacement of one influences the motion of its neighbors. The second is that each individual oscillator has inertia. That is, once it has been set in motion it continues to move until interaction with a neighbor slows it down and reverses its motion. These two conditions, interaction and inertia, are necessary for wave motion.

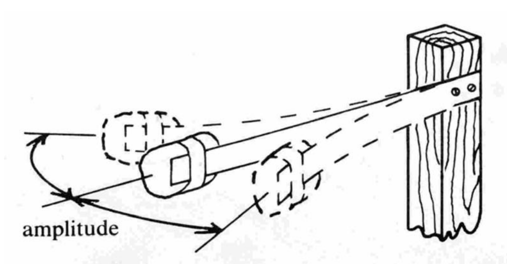


Figure 6·2 An oscillator in motion.

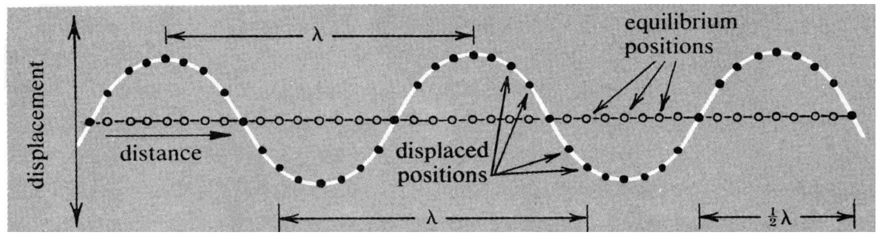


Figure 6.3 Row of oscillators in a medium, showing equilibrium positions and displaced positions in a wave. The wavelength is the distance after which the wave pattern repeats itself.

Equation 6.1

wavelength (meters) = λ

wave number (per meter) $= k$

$\lambda \times k = 1,\ k = \frac{1}{\lambda},\ \lambda = \frac{1}{k}$

EXAMPLES

$\lambda = 0.25$ m

$k = \frac{1}{\lambda} = \frac{1}{0.25\,m} = 4/m$

This is 4 wavelengths/m.

$\lambda = 5.0$ m

$k = \frac{1}{\lambda} = \frac{1}{5.0\,m} = 0.2/m$

This is 0.2 wavelengths/m.

$\lambda = 0.0001$ m $= 10^{-4}$ m

$k = \frac{1}{\lambda} = \frac{1}{10^{-4}\,m} = 10^4/m$

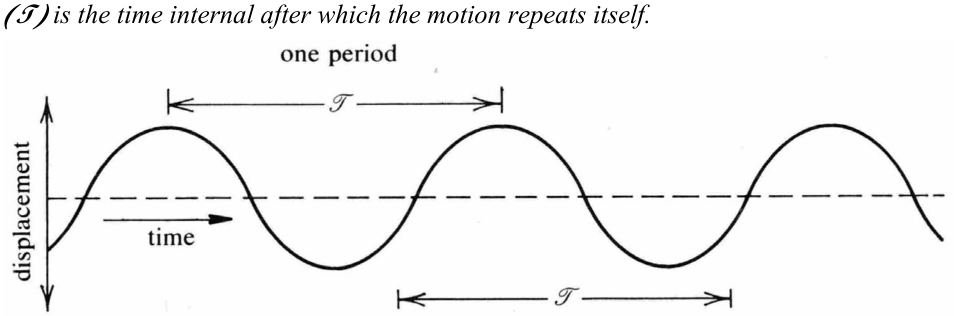
This is 10^4 or 10,000 wavelengths/m.

Wave trains. Look more closely now at the pattern of the oscillators in the medium shown in Fig. 6.3. As a wave travels through the medium, the various oscillators have different displacements at any one instant of time. The wave is represented graphically by drawing a curved line through the displaced positions of all the oscillators (shown above in Fig. 6.3). This curved line, of course, changes as time goes on because the oscillators move. Note, however, that the individual oscillators in the model move only up and down.

Wavelength and wave number. You can see from Fig. 6.3 that the wave repeats itself in the medium. This pattern of oscillators is called a wave train, because it consists of a long train of waves in succession. A complete repetition of the pattern occupies a certain distance, after which the pattern repeats. This distance is called the *wavelength*; it is measured in units of length and is denoted by the Greek letter lambda, **λ**. Sometimes it is more convenient to refer to the number of waves in one unit of length; this quantity is called the *wave number* and it is denoted by the letter **k**. Wavelength and wave number are reciprocals of one another (Eq. 6.1).

Period and frequency. We have just described the appearance of the medium at a particular instant of time. What happens to one oscillator as time passes? It moves back and forth through the equilibrium position as described by a graph of displacement vs. time (Fig. 6.4) that is very similar to Fig. 6.3. The motion is repeated; each complete cycle requires a time interval called the *period* of the motion, denoted by a script "tee," $\mathscr{T}$. The number of repetitions per second is the

Figure 6.4 Graph of the motion (displacement) of one oscillator over time. The period ($\mathscr{T}$) is the time internal after which the motion repeats itself.



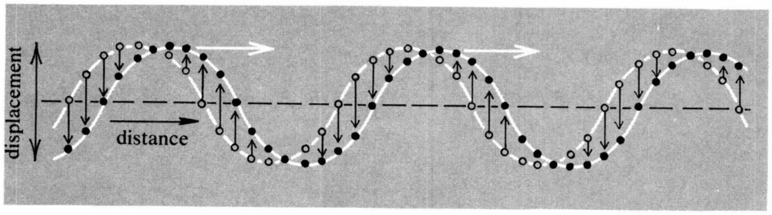
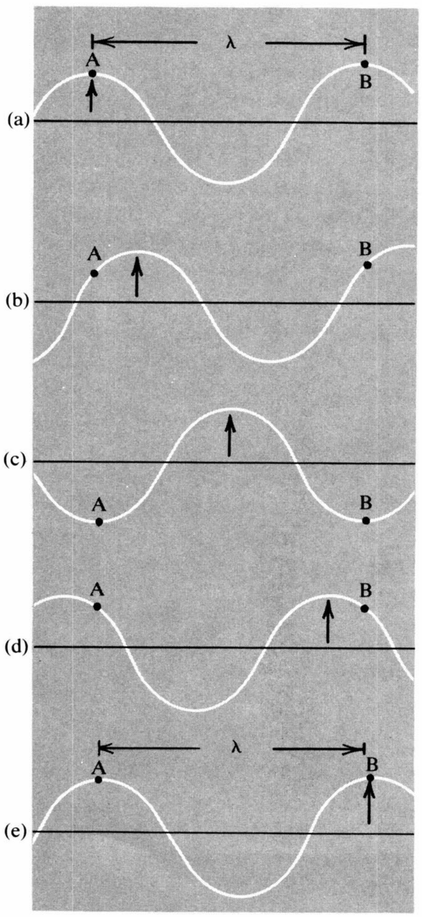


Figure 6·5 The wave moves to the right as the oscillators move up and down. The black circles and black dots represent the displacements of the oscillators at two different times.

Figure 6·6 In one period, oscillators A and B carry out a full cycle of motion from crest to trough and to crest again. The crest initially at A moves to B in this time interval.



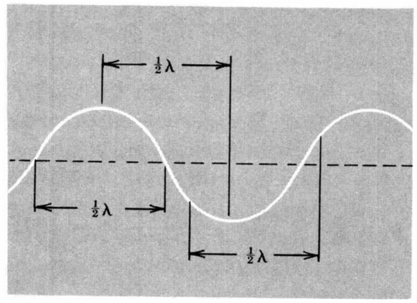
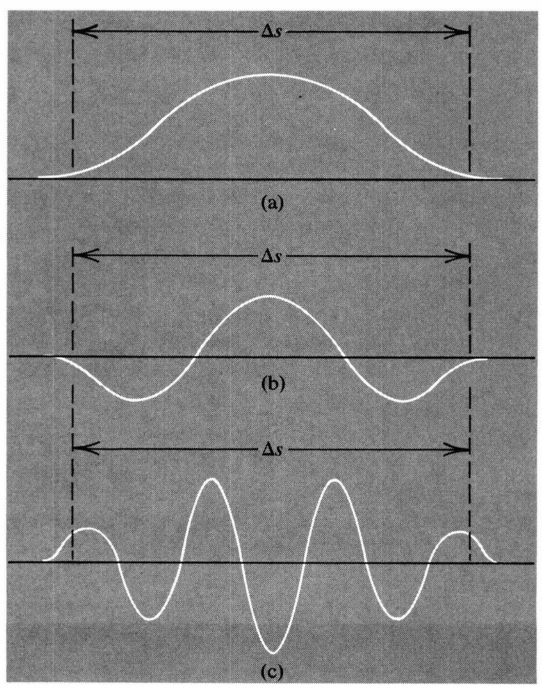


Figure 6·7 Oscillators in a wave train have opposite displacements if their separation is ½ wavelength.

Figure 6·8 Pulse patterns of disturbance in a medium. The approximate length of the pulse is denoted by Δs.



Equation 6.2 (period and frequency of a wave)

period (time for one complete repetition, in seconds) $= \mathcal{T}$
frequency (number of complete repetitions in one second, per second) $=$ f

$$\mathcal{T} \times f = 1,\ f = \frac{1}{\mathcal{T}},\ \mathcal{T} = \frac{1}{f}$$

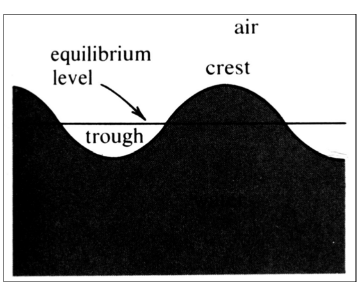
EXAMPLES

If $\mathcal{T} = 0.05$ sec,
$$f = \frac{1}{\mathcal{T}} = \frac{1}{0.05} = 20/\text{sec}.$$
If $\mathcal{T} = 3.0$ sec,
$$f = \frac{1}{\mathcal{T}} = \frac{1}{3.0} = 0.33/\text{sec}.$$
If $\mathcal{T} = 10^{-6}$ sec,
$$f = \frac{1}{\mathcal{T}} = \frac{1}{10^{-6}} = 10^{6}/\text{sec}.$$

Equation 6.3 (wave speed)

wave speed $=$ v

$$v = \frac{\Delta s}{\Delta t} = \frac{\lambda}{\mathcal{T}} \qquad (a)$$
$$v = \lambda f \qquad (b)$$



frequency (symbol f). The period and frequency are reciprocals of one another (Eq. 6.2), just as are the wavelength and wave number. The period and frequency describe the time variation of the oscillator displacements, while the wavelength and wave number describe the spatial variation.

Wave speed. One of the most striking properties of waves is that they give the appearance of motion along the medium. If you look at the pattern of displacements at two successive instants of time (Fig. 6.5), you see that the wave pattern appears to have moved to the right (along the medium), although the individual oscillators have only moved up and down. Since the pattern actually moves, you can measure its speed of propagation through the medium. The wave speed is usually represented by the symbol v (Section 2.2).

You can conduct a thought experiment with the oscillator model for the medium to find a relationship among period, wavelength, and wave speed. Imagine the oscillator at a wave crest carrying out a full cycle of its motion (Fig. 6.6). While this goes on, all the other oscillators also carry out a full cycle, and the wave pattern returns to its original shape. The wave crest that was identified with oscillator A in Fig. 6.6, however, is now identified with oscillator B. Hence the wave pattern has been displaced to the right by 1 wavelength. The wave speed is the ratio of the displacement divided by the time interval (Eq. 2.2), in this instance the ratio of the wavelength divided by the period ($\lambda/\mathcal{T}$, Eq. 6.3a). By using Eq. 6.2, $f = 1/\mathcal{T}$, you can obtain the most useful form of the relationship: $v = \lambda f$, or wave speed is equal to wavelength times frequency (Eq. 6.3b).

Positive and negative displacement. Waves are patterns of disturbances of oscillators from their equilibrium positions. The displacement is sometimes positive and sometimes negative. In Fig. 6.3, the open circles and the horizontal line drawn along the middle of the wave show the equilibrium state of the medium. Displacement upward may be considered positive, displacement downward negative. In water waves, for example, the crests are somewhat above the average or equilibrium level of the water and the troughs are somewhat below the average or equilibrium level of the water. In fact, the water that forms the crests has been displaced from the positions where troughs appear.

By definition, the pattern in a wave train repeats itself after a distance of 1 wavelength. It therefore also repeats after 2, 3, ... wavelengths. Consequently, the oscillator displacements at pairs of points separated by a whole number of wavelengths are equal. If you only look at a distance of 1/2 wavelength from an oscillator, however, you find an oscillator with a displacement equal in magnitude but opposite in direction (Fig. 6.7).

Wave pulses. In the *wave trains* we have been discussing, a long series of waves follow one another, and each one looks just like the preceding one. On the other hand, a ***wave pulse*** is also a disturbance in the medium but it is restricted to only a part of the medium at any one time (Fig. 6.8). It is not possible to define frequency or wavelength for a pulse since it does not repeat itself. The concept of wave speed, however, is applicable to pulses since the pulse takes a certain amount of time to travel from one place to another. In Section 6.2 we will describe how wave trains and wave pulses can be related to one another.

Examples of wave phenomena. The oscillator model for a medium can be applied to systems in which small deviations from a uniform equilibrium arrangement can occur. One such system is a normally motionless water surface that has been disturbed so that water waves have been produced. Another example is air at atmospheric pressure in which deviations from equilibrium occur in the form of pressure variations: alternating higher or lower pressure. Such pressure variations are called sound waves. A third example is an elastic solid such as Jell-O, which can jiggle all over when tapped with a fork. In the oscillator model, movement results from oscillating displacements within the Jell-O after the fork displaced the oscillators at the surface.

Oscillator model for sound waves. Since sound in air is of special interest, we will describe an oscillator model for air in more detail. Visualize air as being made up of little cubes of gas (perhaps each one in an imaginary plastic bag). When acted upon by a sound source, the first cube is squeezed a little and the air inside attains a higher pressure (Fig. 6.9). The first cube then interacts with the next cube by pushing against it. After a while the second cube becomes compressed and the first one has expanded back to and beyond its original volume. The second cube then pushes on the third, and so on. In this way the sound propagates through the air.

The initial pressure increase above the equilibrium pressure may be

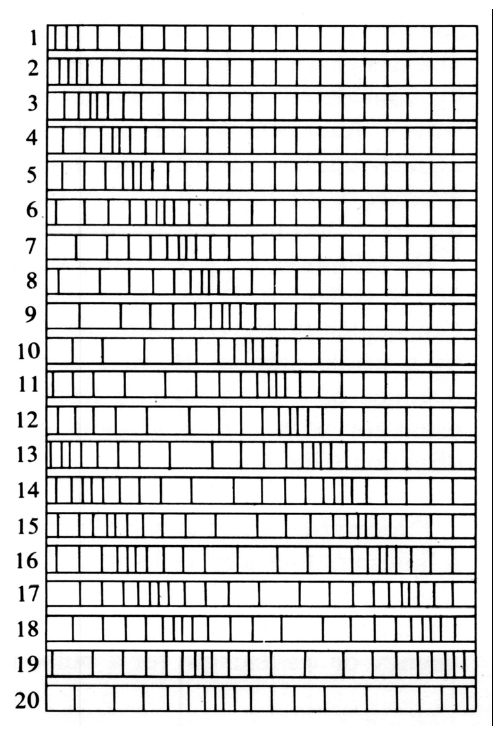


Figure 6.9 A gas bag model for air is used to represent the propagation of a sound wave. An individual bag of gas is alternately compressed and expanded. Its interaction with adjacent bags of gas leads to propagation of the compression and expansion waves.

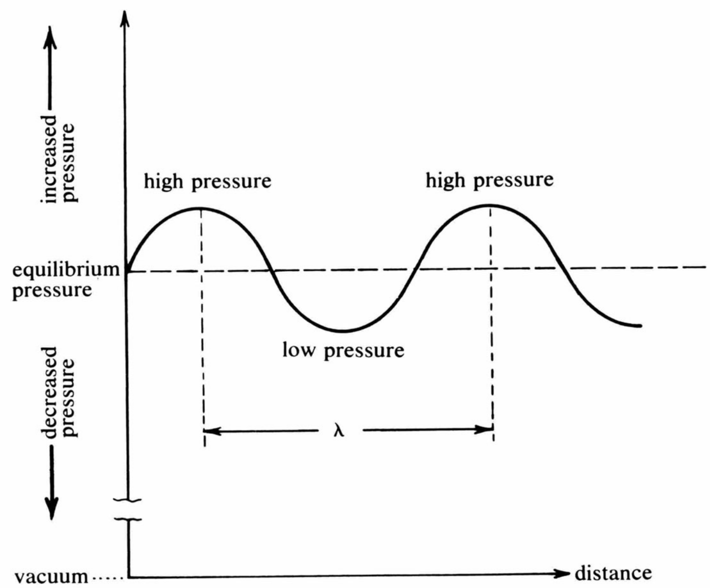


Figure 6.10 Pressure profile in a sound wave. The graph shows deviations from the equilibrium pressure.

created by a vibrating piano string or a vibrating drumhead. In addition to regions of increased pressure, the sound wave also has regions of deficient pressure where the air has expanded relative to its equilibrium state.

Thus the sound wave consists of alternating high-pressure (above equilibrium) and low-pressure (below equilibrium) regions. A pressure profile (pressure versus distance) for a pure tone has the typical wave pattern shown in Fig. 6.10.

6.2 Superposition and interference of waves

The superposition principle. Can you visualize what happens when two waves overlap? In the oscillator model, it is easy to describe the medium at a place where there are two or more waves at the same time. Each oscillator is displaced from its equilibrium position by an amount equal to the sum of the displacements associated with the waves separately (Fig. 6.11). In other words, you visualize the oscillator displacements associated with each of the wave patterns and add them together. This procedure takes for granted that the waves do not interact with one another, but that each propagates as though the others were not present.

The property of non-interaction we have just described is called the *superposition principle*. It makes the combination of waves simple to carry out in thought experiments, and it has been exceedingly valuable for this reason. Fortunately, a wave model that incorporates the superposition principle describes quite accurately many wave phenomena in nature.

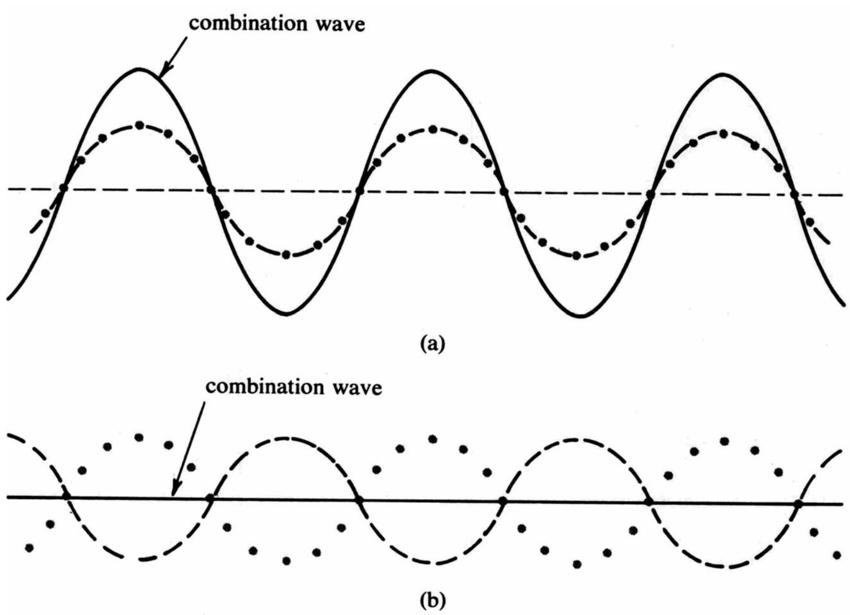


Figure 6.11 Superposition of two waves leads to interference. One wave is represented by black dashes, the other by dots. The combination wave is the sum of both waves and is represented by the solid line.

(a) Constructive interference occurs when dotted and dashed waves reinforce each other.

(b) Destructive interference occurs when dotted and dashed waves cancel each other.

Interference of waves. Consider now what may happen to the oscillator motion as a result of the superposition of two waves. The two waves may combine in various ways. Perhaps each of two wave patterns has an upward displacement of an oscillator at a certain time and at a certain place. In such a case, the upward displacement in the presence of the combined wave will be twice as big as that from one wave alone (Fig. 6.11(a)). If there are simultaneous downward displacements in the two waves separately, the combined displacement will be twice as far down. Suppose you consider a point in space where one wave has an upward displacement and the other wave has an equal downward displacement at the same time. Now, the upward (positive) displacement and the downward (negative) displacement add to give zero combined displacement (zero amplitude of oscillation). In fact, it is possible for

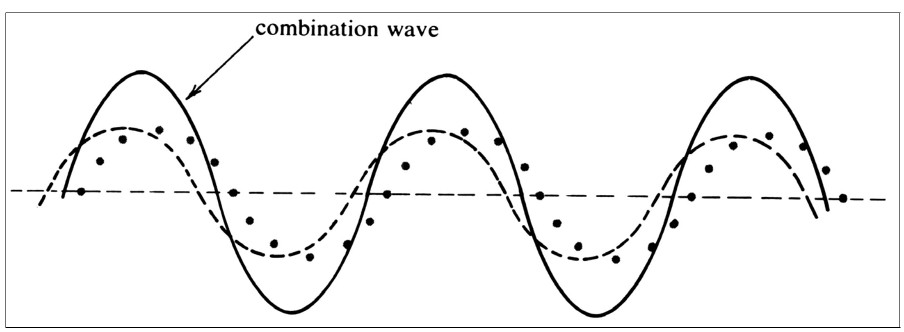
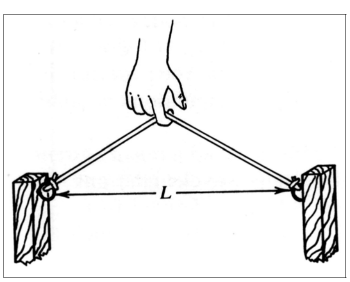


Figure 6.12 Superposition of two waves leading to partially destructive interference. The displacements of the dashed and dotted waves are added together at each point to yield the displacement of the combination wave (represented by the solid line). Note that displacement below the line is negative.

The "one-particle model" for a real object is a "very small object that is located at the center... of the region occupied by the real object." (Section 2.1) This is a way to think about an object so as to focus on the object's position, motion, and inertia without considering its shape and orientation. Complex objects can be thought of as two or more particles that interact in defined ways, or with the many-interacting-particles (MIP) model. In such models, each particle is thought of as a single, tiny bit of matter. The matter itself is thought of as indestructible, or "conserved." Such particles cannot "cancel" one another to cause destructive interference.

A wave, on the other hand, is quite different. A wave, as explained in this chapter, is thought of as a disturbance or oscillation that passes through matter. The displacement of the particles can be positive or negative and, as with (+1) + (–1) = 0, two waves can cancel one another.

In the 20th century, physicists found that matter in the micro-domain behaves in ways that conform with neither the particle nor the wave model. This led to the "wave-particle duality" and quantum mechanics. (Chapter 8).



two waves to combine in such a way that they completely cancel one another, as in Fig. 6.11b.

This characteristic of waves makes their behavior different from what we expect of material objects, particularly when we think of them as single particles (Section 2.1) or as made up of particles. If one particle and another particle are combined, you have two particles, and you cannot end up with zero particles. Two or more waves, however, may combine to form a wave with larger amplitude, a wave with zero amplitude, or a wave with an intermediate amplitude (Fig. 6.12).

This result of the superposition of waves is a phenomenon called *interference*. If waves combine to give a larger wave than either one alone, you have *constructive interference*. If waves tend to cancel each other, you have *destructive interference*. There is a continuum of possibilities between the extremes of complete constructive interference shown in Fig. 6.11(a) and complete destructive interference shown in Fig. 6.11(b). With particles, the concept of destructive interference is meaningless in that the presence of one particle can never "cancel" the presence of another.

Standing waves. When two equal-amplitude wave trains of the same frequency and wavelength travel through a medium in opposite directions, their interference creates an oscillating pattern that does not move through the medium (Fig. 6.13). Such an oscillating pattern is called a *standing wave*. The points in a standing wave pattern where there are no oscillations at all are called *nodes*. At a node, there is always complete destructive interference of the two wave trains; the displacements associated with the two waves at the nodes are always equal and opposite. Because the waves move in opposite directions at the same speed, each node remains at one point in space and does not move; this is the reason behind the choice of name: a *standing* wave does not move.

You can see in Fig. 6.13 that the distance between two nodes must be exactly ½ wavelength. This holds true not only for the illustration but also for *all* standing wave patterns. The reasoning is as follows. At any node, the two wave displacements must always be equal and opposite to produce complete destructive interference. At a distance of ½ wavelength, the displacement associated with each wave has exactly reversed (as illustrated in Fig. 6.7). Thus, the two displacements must again be equal and opposite and again produce a node.

An easy way to set up standing waves is to place a reflecting barrier in the path of a wave. The reflected wave interferes with the incident wave to produce standing waves. The nodes are easy to find because the oscillators remain stationary at a node. This offers a convenient way to determine the wavelength: measure the distance between nodes and multiply by 2.

Tuned systems. It is very fruitful to pursue the standing wave idea one step further. Suppose an elastic rope is tied to a fixed support at each end and the middle is set into motion by being pulled to the side and released (see drawing to left). How will the rope oscillate? To solve this problem, think of the pattern as being made up of wave trains in

Equation 6.4 (Possible number of half wavelengths that fit within L)

$$L=\frac{1}{2}\lambda$$

or

$$L=2\times\left(\frac{1}{2}\lambda\right)=\frac{2}{2}\lambda$$

or

$$L=3\times\left(\frac{1}{2}\lambda\right)=\frac{3}{2}\lambda$$

or

$$L=4\times\left(\frac{1}{2}\lambda\right)=\frac{4}{2}\lambda$$

or

$$L=5\times\left(\frac{1}{2}\lambda\right)=\frac{5}{2}\lambda$$

. . . and so on

Equation 6.5 (wavelengths permitted on a tuned system, from above)

$\lambda=\frac{2}{1}L$, or

$\lambda=\frac{2}{2}L = L$, or

$\lambda=\frac{2}{3}L$, or

$\lambda=\frac{2}{4}L = \frac{1}{2}L$, or

$\lambda=\frac{2}{5}L$

and so on . . .

Equation 6.6 (finding frequency for a given speed and wavelength)

$$f=\frac{v}{\lambda}$$

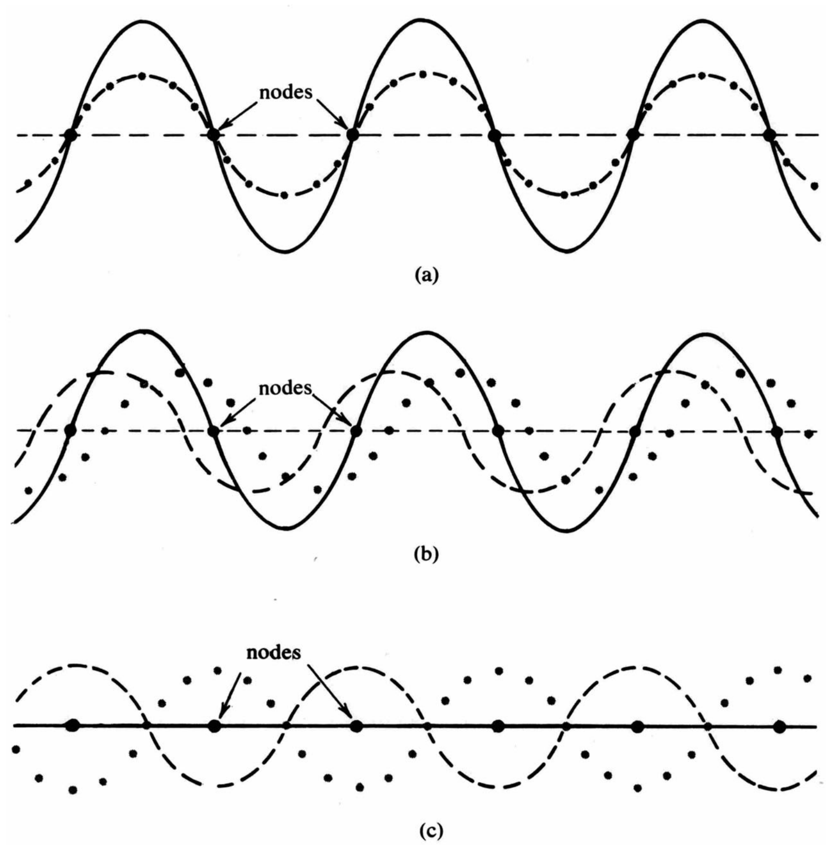


Figure 6.13 The formation of standing waves by the superposition of two wave trains propagating in opposite directions (dotted wave towards right and dashed wave toward left). The combination wave is the solid line. Note the stationary position of the nodes, marked by the large dots.
(a) Constructive interference of the two wave trains.
(b) Partially destructive interference after 1/8th of a period.
(c) Destructive interference after 2/8ths (1/4th) of a period.
Can you draw the pattern after 3/8ths of a period? After 4/8ths (1/2) of a period?

combinations, some moving to the right, others to the left. Because the ends are fixed, the wave pattern must be such that the ends of the rope are its nodes. The length of the rope is the distance between the nodes, which must be an integral multiple of ½ wavelength (Eq. 6.4). It follows that the wavelengths of the waves that can exist on this rope are related to the length of the rope by Eq. 6.5 to satisfy the conditions of nodes at the ends.

A system such as the rope with fixed ends is called a *tuned system*, because it can support only waves of certain wavelengths (Eq. 6.5) and the frequencies related to them by Eq. 6.6 (derived from Eq. 6.3b). The wave speed is a property of the medium from which the tuned system is constructed.

Musical instruments. Musical instruments employ one or more tuned systems whose frequencies are in a suitable relation to one another. For stringed instruments, such as the violin and guitar, the tuned system is a wire or elastic cord; for wind instruments, it is an air column in a pipe closed at one end; for drums, it is an elastic membrane whose edge is fixed; and so on.

The tone of the instrument is determined by the oscillation frequency of the tuned system. It is possible to change the frequency either through changing the length of the tuned system (and therefore changing the wavelength of the allowed standing waves) or through changing the wave velocity by modifying the medium in the tuned system.

Sound waves of a single frequency can be produced in closed pipes of a certain length. Longer pipes produce lower tones. A pressure wave starts at one end of the pipe and travels down the pipe, confined by the walls. When the wave reaches the other end of the pipe, it is reflected back and interferes with waves coming down the pipe. The interference forms a standing wave. This standing wave is of the characteristic wavelength determined by the length of the pipe and has the frequency that we hear.

Equation 6·7

frequencies of the two wave trains (per second) f_1, f_2
time interval (seconds) Δt
number of oscillations N_1, N_2

$$N_1 = f_1\Delta t, \qquad N_2 = f_2\Delta t$$

Equation 6·8

$$\begin{aligned} 1 &= N_1 - N_2 \\ &= f_1\Delta t - f_2\Delta t \\ &= (f_1 - f_2)\Delta t \end{aligned}$$

Equation 6·9

frequency difference Δf

$$\Delta f = f_1 - f_2 \quad \text{(a)}$$

$$\Delta f \Delta t = 1 \quad \text{(b)}$$

EXAMPLE
Frequencies of 255/sec and 257/sec

$$\Delta f = 2/sec$$

$$\Delta t = \frac{1}{\Delta f} = \frac{1}{2/sec} = 0.5\ sec$$

Beats. Standing waves are created by the interference of waves with the same frequency. What will be the combined effect of two waves of differing frequencies? To answer this question, apply the superposition principle in a thought experiment in which two such waves are combined. Suppose the two waves are in constructive interference at one instant of time. Since one wave has shorter cycles than the other before repeating, they will soon get out of step. After a while, the two waves will be in destructive interference, and a little later in constructive interference again. So the net effect is an alternation from constructive interference (loud) to destructive interference (soft) and back again. These alternations in volume are called beats.

It is easily possible to calculate the time interval between two beats from the difference in frequency of the two interfering wave trains. During this time interval the two waves must go from constructive interference to destructive interference and back to constructive interference. Therefore, the higher-frequency wave must vibrate exactly once more than the lower frequency wave. The additional oscillation restores the constructive interference of the two waves, since waves repeat exactly after a whole oscillation. Hence the wave amplitude after the interval is equal to its value before, meaning that the next beat is ready to begin.

The number of oscillations made by either of the two waves is equal to its frequency (oscillations per second) times the time interval ($N_1 = f_1\Delta t$ and $N_2 = f_2\Delta t$, Eq. 6.7). The two numbers, according to the condition, must differ by one ($N_1 - N_2 = (f_1 - f_2)\Delta t$, Eq. 6.8). The conclusion is that the frequency difference times the time interval is equal to one ($\Delta f\ \Delta t = 1$, Eq. 6.9). The frequency of the individual waves determines the overall pitch of the sound, not the beat frequency; in fact, the beat frequency is $f_1 - f_2$.

Wave packets. Standing waves and beats are wave phenomena that are observable when two wave trains are combined. You may, of course, use the superposition principle and the rules for constructive and destructive interference to combine as many different wave patterns as you wish. In the early nineteenth century, it was discovered by Joseph Fourier (1768-1830) that any wave pattern could be formed by a superposition of one or more wave trains, as illustrated below. All wave phenomena can thereby be related to the frequencies, amplitudes, wavelengths, and velocities of the component wave trains in a wave pattern.

To illustrate Fourier's discovery, we will construct a wave pulse close to the one shown in Fig. 6.8a by combining the four wave trains drawn in Fig. 6.14. You are invited to read off the wave amplitudes from the graphs, to add the wave amplitudes of the four waves, and to verify that the combined wave drawn in Fig. 6.14 really is obtained by superposition of the four wave trains. By combining more and more wave trains of other wavelengths and successively smaller and smaller amplitudes, you can achieve further constructive and destructive interference at various locations in the pulse. In this way you could obtain a closer and closer approximation to the wave pulse shown in Fig. 6.8a and Fig. 6.14 (see Fig. 6.15).

The representation of wave pulses by a superposition of wave trains has led to the introduction of the suggestive phrase *wave packet* (instead of wave pulse), which we will also adopt. The superposition procedure can be quite tedious to work out in detail if many wave trains must be combined to achieve success. The essence of the procedure, however, is to select wave trains that interfere destructively in one wing of the wave packet, constructively at the center, and destructively again in the other wing. This can be achieved if one wave train has one more

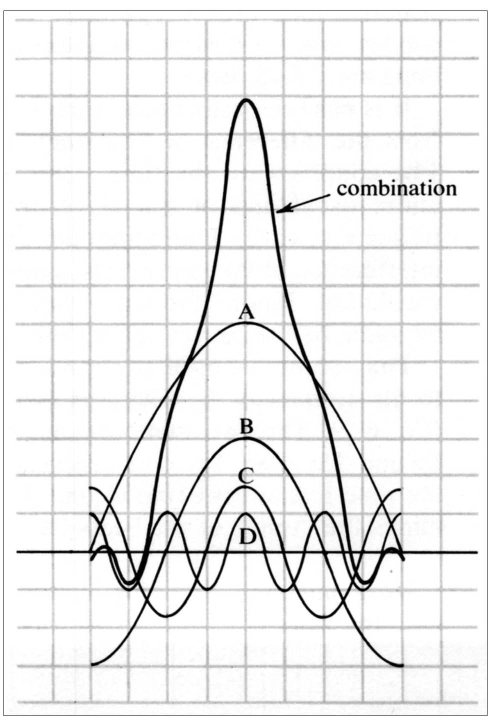
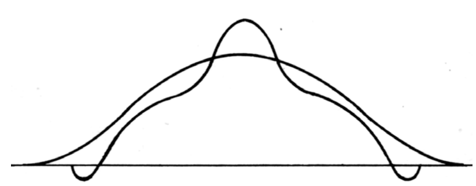


Figure 6.14 (left) The superposition of four wave trains to produce a wave pulse.

Figure 6.15 (below) The wave packet in Fig. 6.14 and the pulse in Fig. 6.8a have been drawn to the same scale for easier comparison.



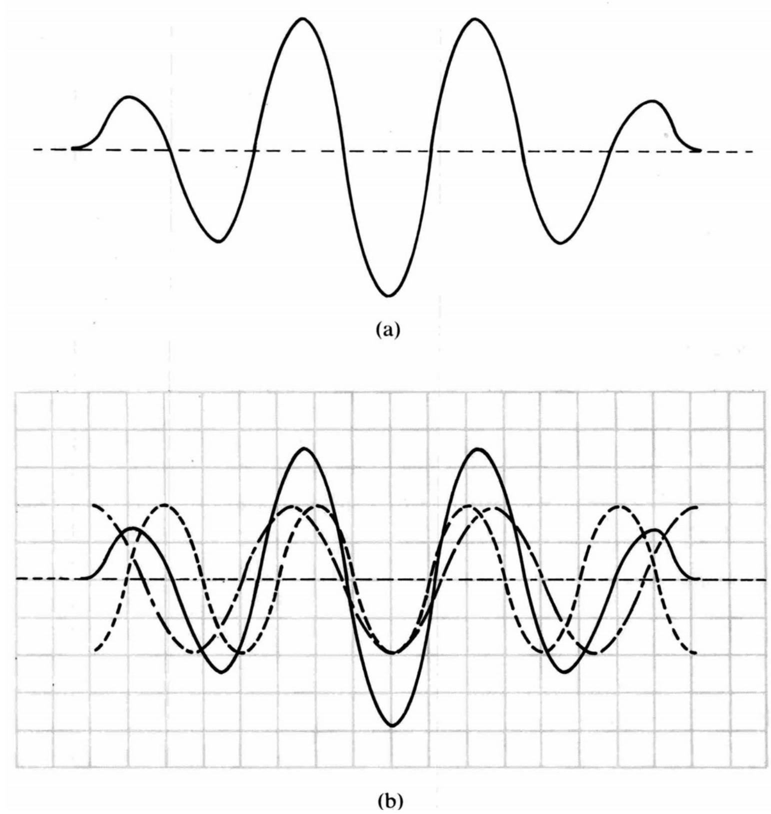


Figure 6.16 The superposition of wave trains to produce a wave packet.
(a) The wave packet pictured in Fig. 68c, enlarged.
(b) A very similar wave packet constructed by the superposition of two wave trains.

full wave over the length of the packet than does the other one. Look, for example, at the wave packet with about four ripples shown in Fig. 6.8c, and reproduced here (Fig. 6.16a). We can combine two wave trains (one with four full waves over the length of the packet and one with three) to find a first approximation to the desired wave packet (Fig. 6.l6b).

Uncertainty principle. We will now formulate a general principle governing the superposition of wave trains to form wave packets. It is called the *uncertainty principle*, and it has played a very important role in the application of the wave model to atomic phenomena, which we will describe in Chapter 8.

Physical significance. The content of the uncertainty principle is that a wave packet that extends over a large distance in space (large Δs) is obtainable by superposition of wave trains covering a narrow range in wave numbers, but that a wave packet that extends over only a short distance in space (small Δs) must be represented by the superposition

Chapter 6: The Wave Theory

PART 2

of wave trains covering a wide range in wave number. It is consequently impossible to construct a wave packet localized in space (small Δs) out of wave trains covering a narrow range in wave number. This idea is known as the "uncertainty principle" because it means that there is an inherent uncertainty in our ability to measure the exact position of a wave packet; Δs represents the "length" of the wave packet and thus the range of uncertainty in our measurement of the packet's position. The size of Δs is closely related to the range of wave numbers included in the packet. We cannot specify the range of wave numbers (Δk) precisely, but we can relate it to the size (Δs) of the wave packet. We will now derive a mathematical model that expresses this relationship.

Equation 6·10

wave number of the two wave trains (per meter) k_1, k_2
wave packet length (meters) Δs
number of waves N_1, N_2

$$N_1 = k_2\Delta s, \qquad N_2 = k_2\Delta s$$

Equation 6·11

$$\begin{aligned} 1 &= N_1 - N_2 \\ &= k_1\Delta s - k_2\Delta s \\ &= (k_1 - k_2)\Delta s \end{aligned}$$

Equation 6·12

wave number difference Δk

$$\Delta k = k_1 - k_2 \quad \text{(a)}$$

$$\Delta k\Delta s = 1 \quad \text{(b)}$$

Mathematical model. The calculation proceeds in the same way as the calculation for the time interval between beats in Eqs. 6.7, 6.8, and 6.9. First, we select two wave trains with different wave numbers k_1 and k_2, one a little larger and one a little smaller than the average wave number of all the waves needed. Each wave train has a certain number of waves ($N_1 = k_1\Delta s$ and $N_2 = k_2\Delta s$) within the length (Δs) of the wave packet (Eq. 6.10). By how much do these two numbers have to differ? They have to differ sufficiently so that the two wave trains are in destructive interference in the regions to the left and to the right of the wave packet's center, where they are in constructive interference. The distance between the two regions is approximately the spatial length Δs of the wave packet. Now, to achieve the desired destructive interference in both regions, the wave train with the shorter wavelength has to contain at least one more whole wave than the other in the distance Δs, that is: $1 + N_2 = N_1$, or $1 = N_1 - N_2$. This condition is applied in Eq. 6.11 to yield an important result: the range of wave numbers (Δk) times the width of the wave packet (Δs) is equal to one (Eq. 6.12b).

Comparison of beats and wave packets. It is clear that Eqs. 6.9 and 6.12b are closely similar. You may consider both of them as statements of an uncertainty principle for wave packets if you are willing to think of one beat pulsation as a wave packet. Equation 6.12 refers to the size of the wave packet in space. Equation 6.9 refers to the duration of the wave packet in time. The wave trains included in a wave packet have a certain average wave number or frequency, and extend above and below these average values by an amount equal to about one half of the wave number difference Δk or frequency difference Δf. The wave packet includes wave trains of substantial amplitude within this range of wave number or frequency, and wave trains of progressively smaller and smaller amplitude outside this range. The exact amplitude distribution of the included wave trains is determined by the shape of the wave packet and can be calculated by more complicated mathematical procedures developed by Fourier and later workers. We apply the uncertainty principle to wave packets below in Examples 6.1 and 6.2

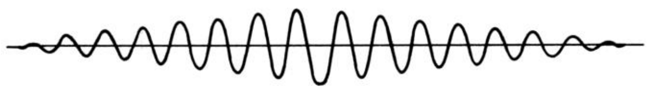
EXAMPLE 6.1. A telegraph buzzer operates at a pitch of 400 vibrations per second. A sound wave packet is formed by depressing the key for 0.1 second. What is the frequency range in the wave packet?

Solution:

$\Delta f\,\Delta t = 1$, $\Delta t = 0.1$ sec., hence $\Delta f = (1/\Delta t) = (1/0.1 \text{ sec}) = 10/\text{sec}$.

The frequency range is about 395 per second to 405 per second.

EXAMPLE 6.2. The wave packet pictured here is 0.08 meter long and contains approximately 16 ripples. What is the wave number range in this wave packet?



Solution : Average wave number k $= \frac{16}{0.008} = 200/m$

$\Delta k\, \Delta s = 1$, $\Delta s = 0.08\ m$, $\Delta k = \frac{1}{\Delta s} = \frac{1}{0.08m} = 12/m$

The wave number range is 194/m to 206 /m

6.3 Huygens' Principle

Ripple tank. Let us now return to study the propagation of waves by experimenting with water waves. A ripple tank is a useful device for observing water waves. It is a shallow tank with a glass bottom through which a strong light shines onto a screen (Fig. 6.17). Dipping a wire or paddle into the water, creates waves on the water surface; the crests of the waves create bright areas on the screen and troughs create shadows. The patterns of disturbance of the water surface may be observed (Fig. 6.18). A wide paddle generates straight waves (Fig. 6.18a), while the point of a wire generates expanding circular waves (Fig. 6.18b).

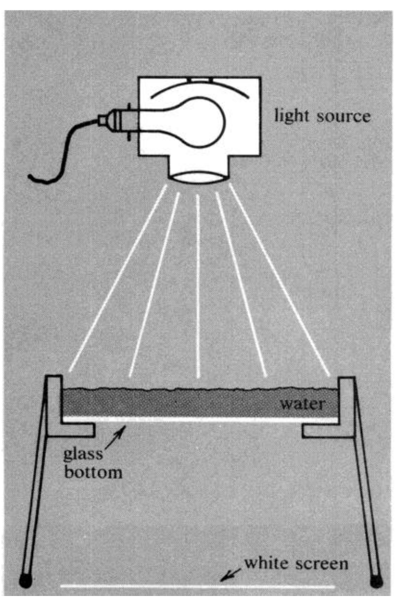


Figure 6.17 (left) Diagram of a ripple tank used for the production and observation of water waves. The wave crests and troughs create bright areas and shadows on the screen.

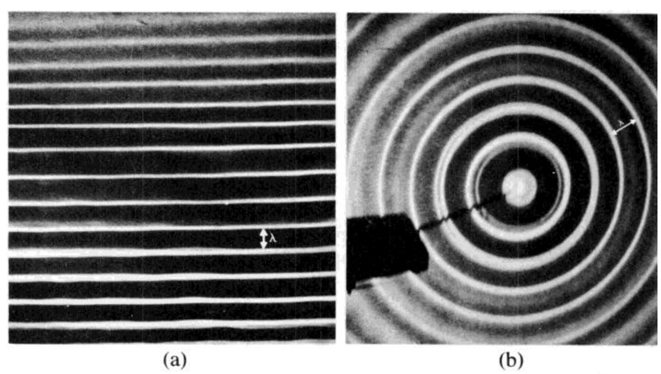


Figure 6.18 (above) Water waves in a ripple tank. (a) Waves generated by a wide paddle. (b) Waves generated by the point of a wire.

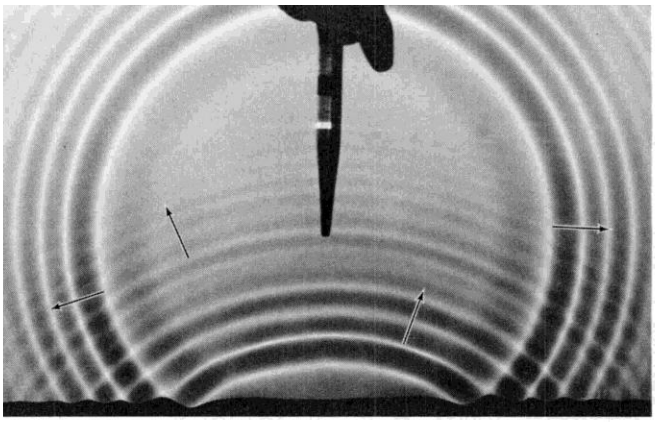


Figure 6.19 The bright lines of the wave crests indicate the wave fronts. The arrows at right angles to the wave fronts indicate the direction of propagation. The waves were originally produced by the tip of the pointer at the center of the photo. The wave fronts form circles centered on the point where they were created until they reflect from the barrier at the bottom of the photo. Where does the wave appear to be diverging from after it is reflected? Can you relate this to what you see in a plane (flat) mirror, as in Fig. 5.17?

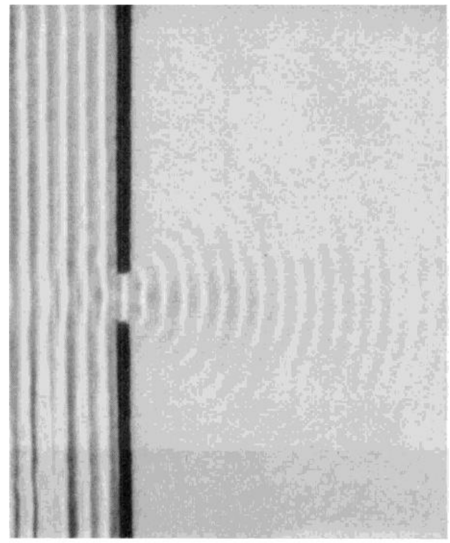


Figure 6.20 Straight waves from the left impinge on a barrier with a hole. Note the curved, circular shape of the wave front to the right of the barrier

The point of a wire may be considered as a point source of waves. The reflection of a circular wave pulse created by a pencil point touched to the water surface is shown in Fig. 6.19.

Wave patterns. To describe the pattern, we identify the *wave front*, which is the line made by each wave crest or trough, and the *propagation direction* in which the wave is traveling. The wave always travels in the direction at right angles to the wave front. Therefore, the wave travels in different directions at different parts of a curved wave front such as the one shown in Fig. 6.19.

You can make an interesting discovery if you use a barrier to block off all but one small section of the water. The waves passing through the hole from one side of the barrier to the other spread out in ever increasing circles (Fig. 6.20). This shows a very important result: the small section of the wave front acts as if it were itself a point source of waves.

Huygens' wavelets. In the oscillator model, the oscillator in the small section moves in rhythm with the waves impinging from the source side of the barrier; it also interacts with the oscillators on the other side and

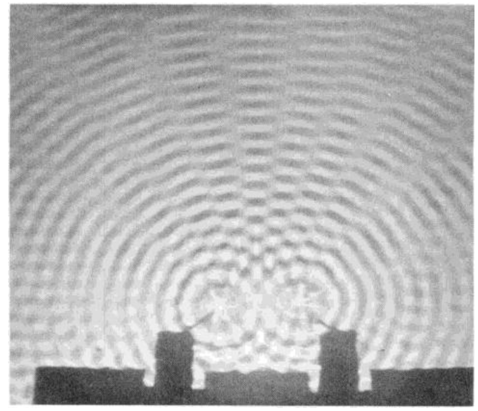


Figure 6.21 Two point sources produce an interference pattern. Note the lines of "nodes" fanning out from the sources.

sets them in motion as though it were a point source. In fact, you can think of every point of a wave front as the source of *wavelets* (numerous mini-waves generated by another wave) that radiate out in circles. That is, each oscillator interacts equally with the other oscillators in all directions from it. This principle is called *Huygens' Principle*. The wavelets have the same frequency of oscillation as their source points in the old wave front. When a wave front encounters a barrier, then most parts of the wave front are prevented from acting as wave sources. What remains is the circular wavelet originating from that part of the wave front that passes through the hole in the barrier.

Two-hole interference. When the barrier has two holes, the waves not only pass through both holes and spread out, but there also is interference between the waves coming from these two "sources." The observable result is very similar to the interference produced by waves from two adjacent point sources (Fig. 6.21). Note the lines of "nodes" fanning out at various angles from the sources, forming what is known as a "two-hole (or double-slit) interference pattern." This pattern demonstrates the existence of interference and can be observed in all waves (including light and sound), not just those in a ripple tank.

Construction of wave fronts. The position of the wave front at successive times may be found by seeking the region of constructive interference of the wavelets emanating from all the source points in a wave front. When there is no barrier, the complete circular wavelets originating from each point in the wave front are not seen because of destructive interference among them.

Schematic diagrams for the procedure of locating the constructive interference are drawn in Fig. 6.22. These diagrams show a wave crest at three successive instants. Huygens' Principle is applied to source points a in the initial wave crest AB to obtain the circles b, c, d. The destructive and constructive interference of all these wavelets results in a new wave crest at the position of the common tangent line CD of all the circles. After a second equal time interval, all the circles

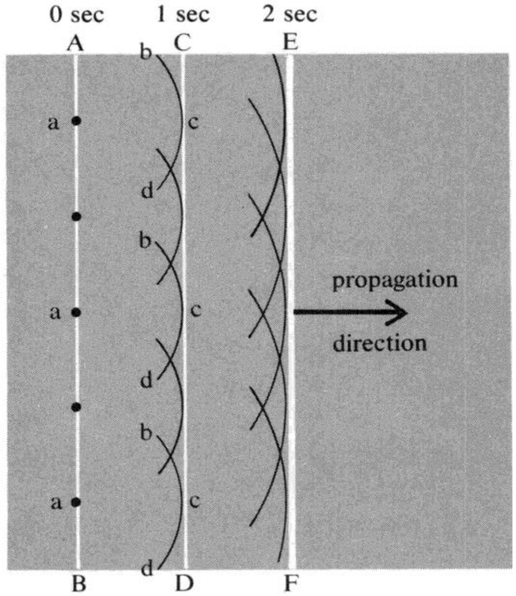


Figure 6.22 The wave front AB (thick white line) advances to CD and then EF, which are the common tangent lines of all the circular wavelets (thin black lines) from Huygens' sources (black dots) in the wave front at AB.

are twice as large, but again the interference effects result in a wave crest EF at the position of the common tangent line of all the larger circles. In this way the straight wave crest advances.

6.4 Diffraction of waves

It is clear from Fig. 6.20 that waves do not necessarily travel in straight lines. Even though the incident wave is headed to the right, the wave transmitted through the hole has parts that travel radially outward

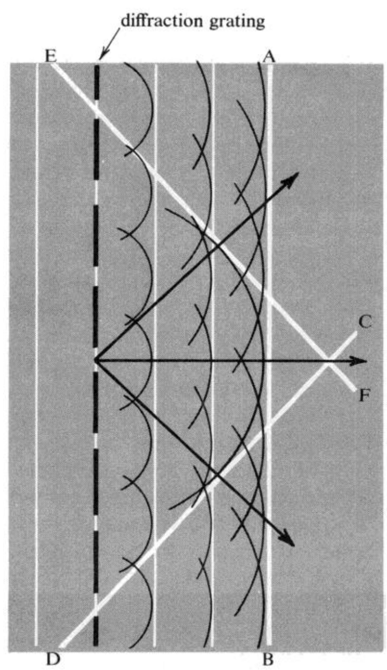


Figure 6.23 Huygens' Principle is used to find the waves transmitted by a diffraction grating. Note the wavelets (thin, curved black lines) centered on the slits. The white lines indicate the undiffracted wave crests (along common tangent line AB) and the diffracted wave crests (common tangent lines CD and EF). The black arrows show the directions of propagation of the observable waves.

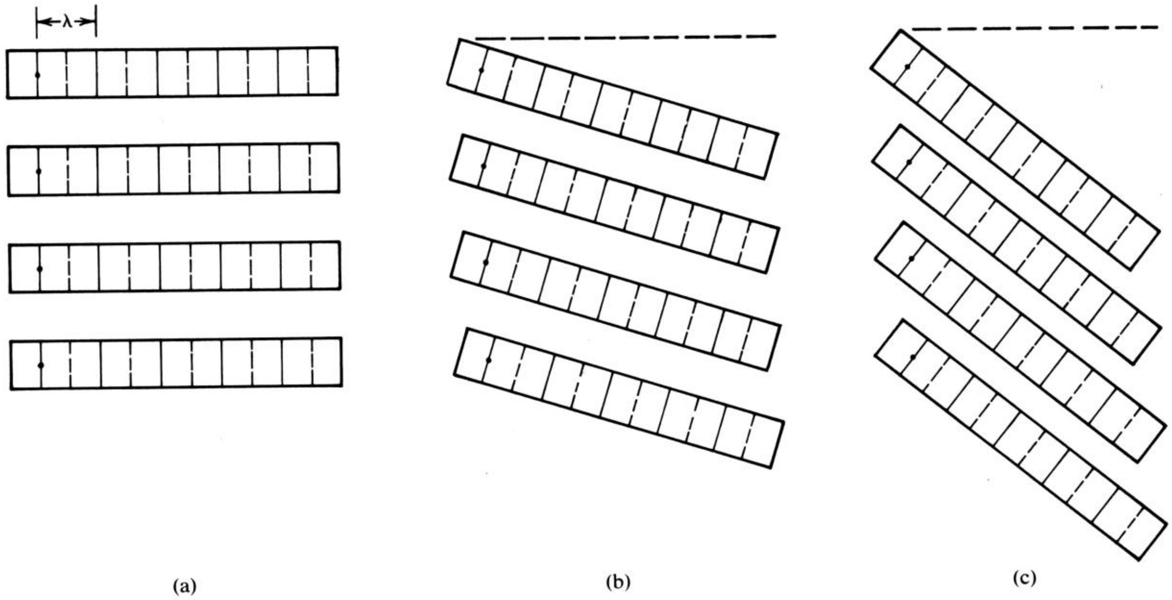


Figure 6.24 Paper strip analogue model for diffraction of waves by a grating. Four paper strips are marked at equal intervals to represent wave troughs and crests. The four strips are then pinned in a row to represent four wave trains passing through equidistant slits in a grating. The strips may be rotated, but are always kept parallel so that the strip direction represents the propagation direction. Interference is determined by the superposition of crests and troughs on the strips.
(a) Constructive interference in the un-diffracted direction is indicated by the alignment of crests with crests, troughs with troughs.
(b) Destructive interference is indicated by the alignment of the crests of one "wave" and the troughs of the adjacent "wave."
(c) Constructive interference in the diffracted direction is indicated by the alignment of crests with crests, troughs with troughs.

from the hole. In other words, the wave was deflected (or bent) by the barrier. This process of deflection of waves passing beside barriers is called *diffraction*. Diffraction makes it possible for waves to bend around a barrier.

Diffraction grating. Let us now apply Huygens' Principle to a device called a *diffraction grating*. A diffraction grating has many evenly spaced slits through which waves can travel. Between the slits, waves are absorbed or reflected. A wave coming through the slits radiates out from each slit in circular wavelets according to Huygens' Principle (Fig. 6.23). You do not observe simple circular waves, however, because the many waves interfere, sometimes constructively and sometimes destructively. A convenient analogue model for diffraction that can be constructed from four strips of paper is described in Fig. 6.24.

(a)

(b) [Enlarge-ment of (a)]

(c) [Enlargement of (b)]

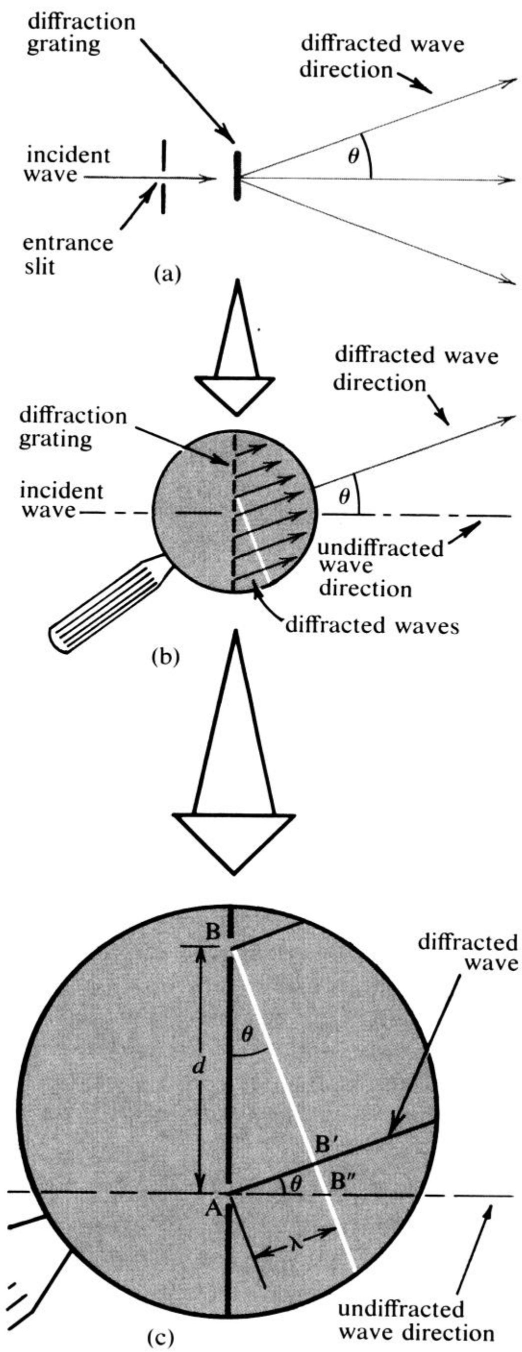


Figure 6·25 Construction of a mathematical model for diffraction by a large diffraction grating.
(a) Waves impinge on the grating from the left. Part of the wave pattern is diffracted at an angle θ, part continues in the undiffracted direction.
(b) Enlarged view of the grating shows that waves passing through adjacent slits travel different distances to contribute to the same wave front (white line).
(c) Constructive interference of diffracted waves occurs if the wave trains from adjacent slits are exactly 1 wavelength out of step as they contribute to one wave front (white line).

distance between slits	d
wavelength	λ
diffraction angle for constructive interference	θ
additional path length for waves passing slit A compared to waves passing slit B	$\overline{AB}'$

Diffraction condition: $\overline{AB}' = \lambda$

Step I: By definition, $\sin \angle ABB' = \overline{AB}'/\overline{AB} = \lambda/d$.
Step II: Prove $\angle ABB' = \theta$.
(i) Extend line BB′ to the undiffracted wave direction at B″.
(ii) θ ***is complementary to*** $\angle AB''B'$ ***in right triangle*** $AB''B'$.
(iii) $\angle ABB'$ ***is complementary to*** $\angle AB''B'$ ***in right triangle*** $AB''B$.
(iv) Hence $\angle ABB' = \theta$.
Step III: It follows from I and II that $\sin \theta = \lambda/d$.

With a large diffraction grating of many slits (perhaps 10,000 slits or more), constructive interference of the waves from all the slits occurs only when the adjacent strips are exactly one, two, or three waves out of step. For all other directions, you can find pairs of close or distant slits that give complete destructive interference and thereby cancel one another's wavelets. Waves are therefore diffracted by the grating only

Equation 6.13
(diffraction grating)

distance between slits (meters) $= d$
diffraction angle $= \theta$

$$\text{sine } \theta = \frac{\lambda}{d}$$

into certain special directions. The diffraction angle can be calculated from the condition for constructive interference (Fig. 6.25).

The diffraction grating formula states that the sine of the angle of diffraction is equal to the ratio of the wavelength to the distance between slits. (See Eq. 6.13 and Example 6.3.) The most important practical application of the diffraction grating has been to the study of light, which will be described in the next chapter.

EXAMPLE 6·3. **Use of the diffraction formula.**

(a) $\lambda = 0.2$ m, $d = 0.3$ m, $\theta = ?$

$$\text{sine } \theta = \frac{\lambda}{d} = \frac{0.2\text{m}}{0.3\text{m}} = 0.67$$
$$\theta = 42°$$

(b) $d = 10^{-6}$ m, $\theta = 25°$, $\lambda = ?$

$$\text{sine } 25° = 0.42$$
$$\lambda = d \text{ sine } \theta = 10^{-6} \text{ m} \times 0.42 = 0.42 \times 10^{-6} \text{ m}$$

(c) $\lambda = 10^3$ m, $\theta = 15°$, $d = ?$

$$\text{sine } \theta = 0.26$$
$$d = \frac{\lambda}{\text{sine } \theta} = \frac{10^3\text{m}}{0.26} = 3.9 \times 10^2 \text{ m}$$

(d) $\lambda = 10^{-4}$ m, $d = 10^{-2}$ m, $\theta = ?$

$$\text{sine } \theta = \frac{\lambda}{d} = \frac{10^{-4}}{10^{-2}} = 10^{-2}$$
$$\theta = 0.6°$$

(e) $\lambda = 0.3$ m, $d = 0.2$ m, $\theta = ?$

$$\text{sine } \theta = \frac{\lambda}{d} = \frac{0.3}{0.2} = 1.5$$

θ does not exist.

Diffraction by single slits and small obstacles. Huygens' Principle can also be applied to diffraction by a single slit opening (Fig. 6.20) and to diffraction by a short barrier. The result of the theory suggests that the ratio of the wavelength to a geometrical dimension of the diffracting barrier is of decisive importance for diffraction. In fact, if this ratio is very small (short wavelength, large slit, or large obstacle), the angles of diffraction are very small, so that diffraction is hardly noticeable. If the ratio is large (long wavelength, small slit, or small obstacle),

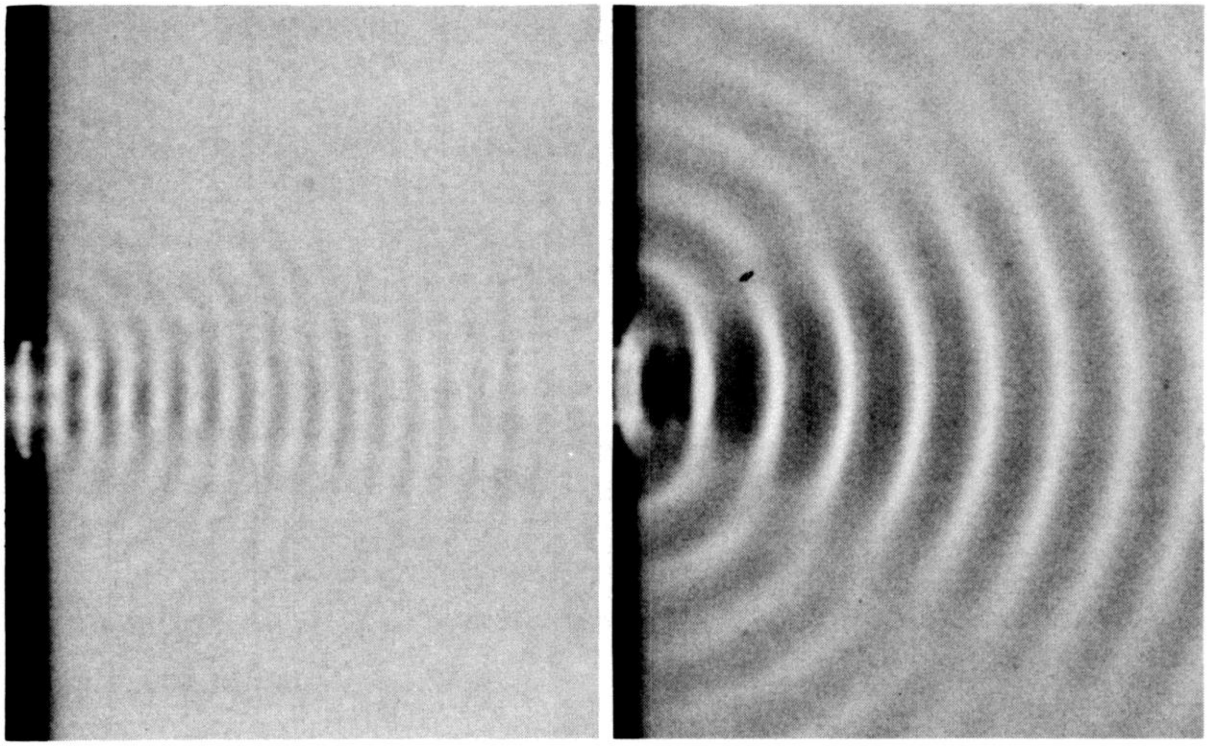
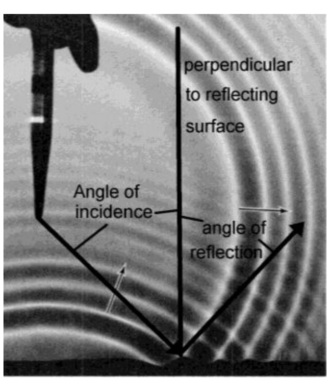


Figure 6.26 Diffraction of waves by an opening. In both photos, the waves are moving from left to right. In the left photo, the wavelength is relatively short (1/3 the width of the opening), so there is little diffraction, and only very weak waves are diffracted away from the original direction of propagation. In the right photo, the wavelength is longer (2/3 the width of the opening), and the waves experience substantial diffraction, spreading out in all directions. After passing through the opening, the wave fronts are essentially semicircles, showing that the waves are now moving in various directions away from the opening. This is a practical demonstration of Huygen's Principle: the waves passing through the opening act as point sources of new waves, which then travel in all directions away from the sources.

then diffraction covers all angles, but the amplitudes of the diffracted waves are very small because the slit or obstacles are small. For intermediate values of the ratio (wavelength comparable to the slit or obstacle in size), diffraction is an important and easily noticeable phenomenon. Two photographs of waves in a ripple tank (Fig. 6.26) show long and short wavelength waves passing through an opening and being diffracted when they pass through an opening. The greater diffraction of the longer wavelength waves is obvious.



6.5 Reflection of waves

The ripple tank photograph to the left (from Fig. 6.19) shows reflection of an expanding circular wave packet. We picked one point on the barrier and drew arrows showing the approximate direction of propagation before and after reflection from that point. The angles of incidence and reflection (as defined in Fig. 5.11) are shown. You can measure the angles to test whether they are equal; we measured one to be 43.5° and the other to be 45°; this is satisfactory agreement given the accuracy of our measurements.

We can also use Huygens' Principle to investigate the relation of these angles in a more general way. According to this principle, each point in a wave front acts like a source of wavelets propagating outward. The wavelets have the same frequency and wavelength as the original waves. The common tangent line of the wavelets is the wave front they produce by constructive interference.

The reflection process is illustrated in Fig. 6.27. A straight wave is incident on the reflecting barrier obliquely from the left. Between the

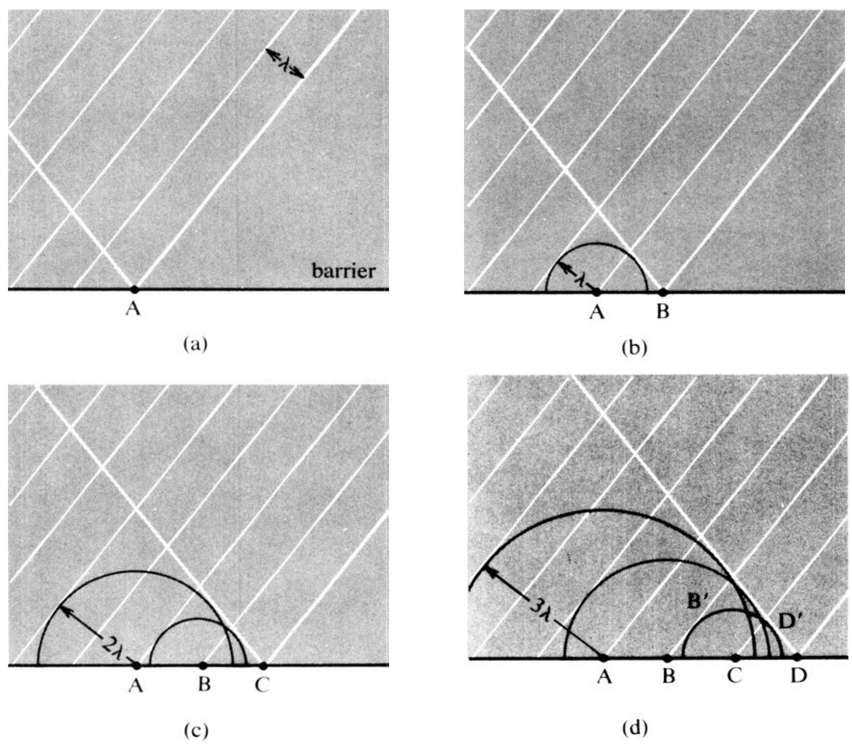


Figure 6.27 Reflection of waves by a barrier. The incident wave crests (white lines) are advancing toward the lower right. Only one reflected wave, moving toward the upper right, is shown. To avoid clutter in the diagram, we have not drawn the other reflected waves.

(a) Three wave crests are striking the barrier, which has reflected a section of the first crest. Point A on the first crest acts as a source of Huygens' wavelets.

(b) The wave crests advance by a distance of one wavelength (λ), and the wavelet from Point A has expanded into a semicircle of radius λ. Point B becomes a source of wavelets.

(c) The wave crests advance by another wavelength; the wavelet from Point A now has a radius of 2λ; the wavelet from B has a radius λ. and the Point C becomes a source of wavelets.

(d) The wavelet from A has radius 3λ; the wavelet from B has radius 2λ, the wavelet from C has radius λ, and Point D becomes a source of wavelets.

The wavelets constructively interfere all along the common tangent line DD', which defines the location and direction of the reflected wave fronts.

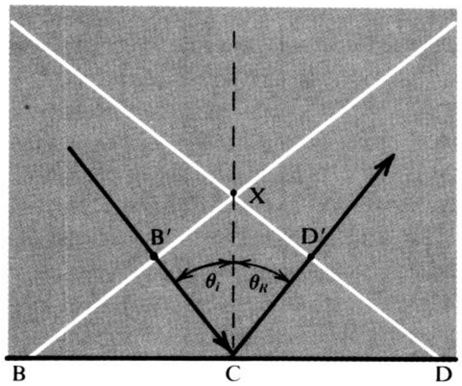


Figure 6·28 Construction of a mathematical model for wave reflection based on Fig. 6·27. Arrows B'C and CD' represent, respectively, the incident and reflected propagation directions. They are at right angles to the corresponding wave fronts BB' and DD' (white lines). Consider the two right triangles XCB' and XCD'. They share the common hypotenuse XC and have two sides equal, CB' = CD' = λ. Hence the two triangles are congruent. It follows that corresponding angles are equal, $\theta_i = \theta_R$.

four successive instants shown in Fig. 6.27, the wave advances between each drawing by 1 wavelength. The Huygens wavelets formed by the first wave crest passing through points A, B, C, and intermediate points on the barrier have a common tangent, which is the reflected wave front. The crests of the Huygens' wavelets all fall on the common tangent, where they interfere constructively; at all other points, the wavelets interfere destructively and cancel one other.

Equation 6.14 (Law of Reflection)

angle of incidence = θ_i

angle of refraction = θ_R

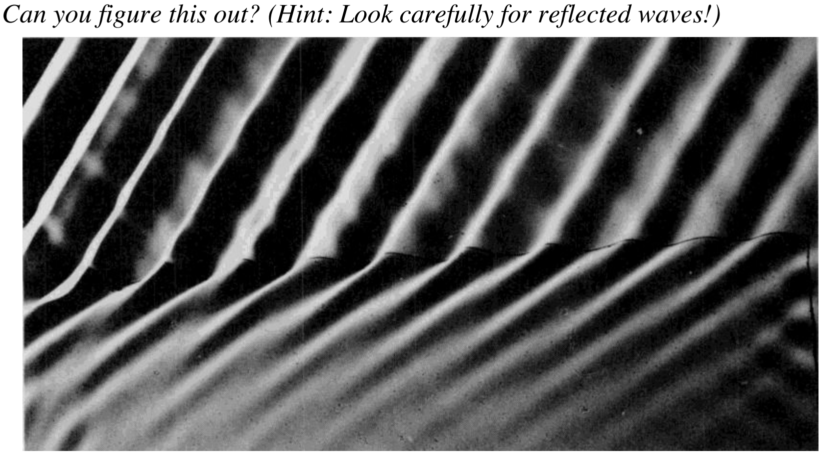
$$\theta_i = \theta_R$$

To relate the angles of incidence and reflection, the directions of propagation have to be taken into account. This is done in Fig. 6.28, where only one incident and one reflected wave crest from Fig. 6.27d are included. The application of Huygens' Principle in Fig. 6.28 results in a familiar conclusion: the angle of incidence is equal to the angle of reflection (Eq. 6.14). This statement may be called the *law of wave reflection*.

6.6 Refraction of waves

When a wave propagates from one medium into another, its direction of propagation may be changed. An example of this happening with water waves is shown in Fig. 6.29. The boundary here is between deep water above and shallow water below. Even though water is the

Fig 6.29 Water waves passing from a deeper region to a shallower region are refracted and travel in a different direction at the boundary. Huygens' Principle does not reveal which direction the waves are traveling. Can you figure this out? (Hint: Look carefully for reflected waves!)



Equation 6.3b

$v = f\lambda$

material on both sides of the boundary, it acts as a different medium for wave propagation when it has different depths. You can see that the wavelength is shorter in the shallow water and can infer from this that the wave speed is slower there (Eq. 6.3b).

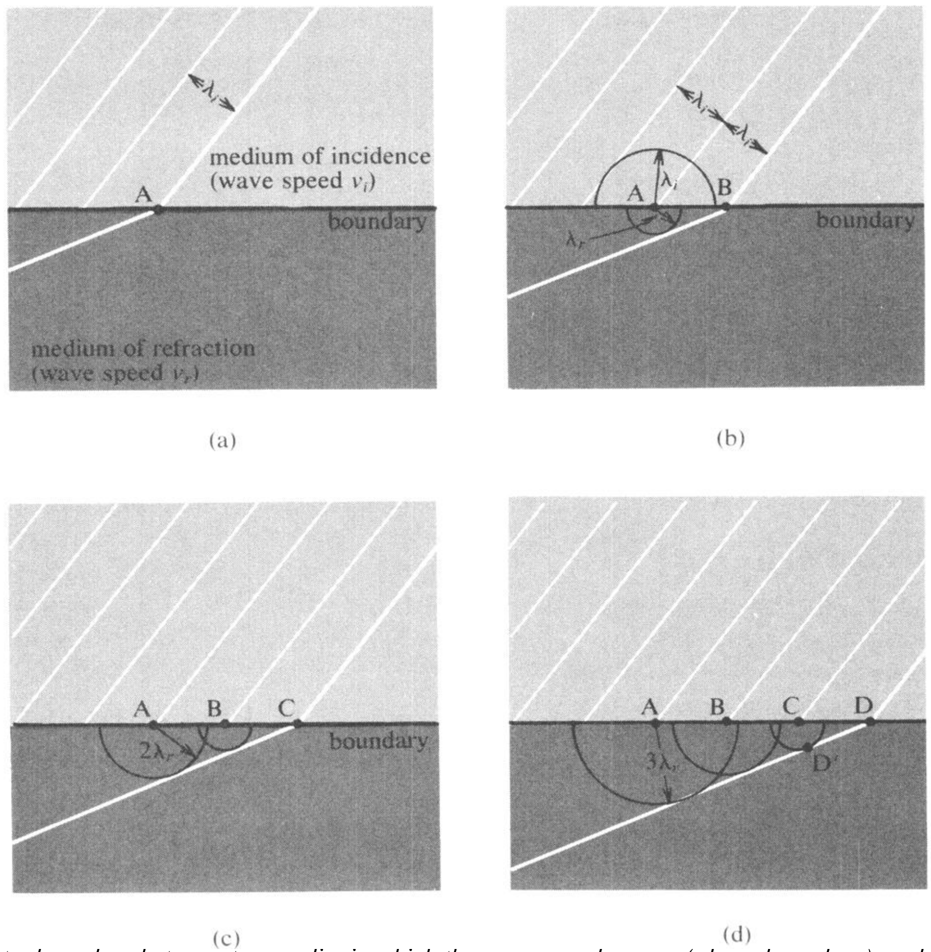


Figure 6.30 Refraction at a boundary between two media, in which the wave speeds are v_i *(above boundary) and* v_r *(below boundary). The wave speed is assumed to be* ***less*** *below the boundary than above it. (*v_r *is* ***less*** *than* v_i*). The waves are moving downward to the right.*

(a) Three wave crests (white lines) are incident on the boundary, which has refracted a section of the waves. Point A on the first crest acts as a source of Huygens' wavelets.

(b) The wavelet from point A has expanded into a semicircle of radius λ_r *below the boundary and a semicircle of radius* λ_i *above the boundary. The latter gives rise to a reflected wave (see Fig. 6.27) and will not be described further. The wave crests advance by the distance* λ_i*. Point B becomes a source of wavelets.*

(c) The wavelet from A has reached radius $2\lambda_r$*, the wavelet from B has radius* λ_r*. Point C becomes a source of wavelets.*

(d) The wavelet from A has radius $3\lambda_r$*, the wavelet from B has radius* $2\lambda_r$*, and the wavelet from C has radius* λ_r*. The common tangent line DD' coincides with the refracted wave front. Note the change in the direction of propagation (which is perpendicular to the wave front): we can see from the diagram that the refracted wave (below the boundary) is traveling* ***slower*** *and in a direction* ***farther*** *from the boundary surface than the incident wave (that is,* ***closer*** *to the perpendicular to the boundary).*

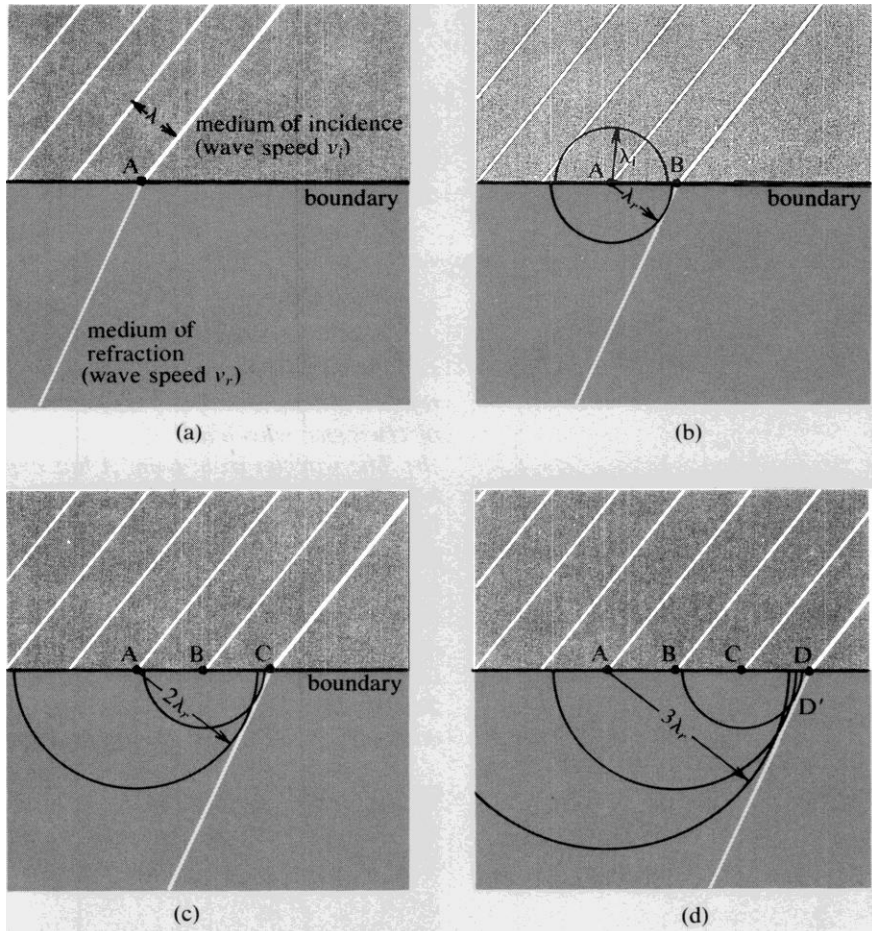


Figure 6.31 Refraction at a boundary between two media, in which the wave speeds are v_i (above boundary) and v_r (below boundary). The wave speed is assumed to be ***greater*** *below the boundary (v_r is* ***greater*** *than v_i). (This is similar to Figure 6.30 but the speeds are reversed.) The waves are assumed to be moving downward to the right.*

(a) Four wave crests (white lines) are incident on the boundary, which has refracted a section of the waves. Point A on the first crest acts as a source of Huygens' wavelets.

(b) The wavelet from point A has expanded into a semicircle of radius λ_r below the boundary and a semicircle of radius λ_i above the boundary. The latter gives rise to a reflected wave (see Fig. 6.27) and will not be described further. The wave crests advance by the distance λ_i. Point B becomes a source of wavelets.

(c) The wavelet from A has radius $2\lambda_r$; the wavelet from B has radius λ_r. Point C becomes a source of wavelets.

(d) The wavelet from A has radius $3\lambda_r$, the wavelet from B has radius $2\lambda_r$, and the wavelet from C has radius λ_r. The common tangent line DD' coincides with the refracted wave front. Note the change in the direction of propagation (which is perpendicular to the wave front): we can see from the diagram that the refracted wave (below the boundary) is traveling ***faster*** *and in a direction that is* ***closer*** *to the boundary surface than the incident wave (that is,* ***further*** *from the perpendicular to the boundary).*

Note: Although we have assumed above that the waves are traveling toward the right, this demonstration can also be carried out using the same diagram with the waves traveling in the opposite direction. Thus wave theory based on Huygens' Principle predicts that refracted waves will follow the same path in either direction. Does this seem reasonable to you? Can you suggest any observations or experiments that would confirm or refute this?

The refracting boundary. The change in the direction of propagation is called refraction, the same term that was introduced in Section 5.2. We will now find the law of refraction of waves by applying Huygens' Principle to the propagation of the wave across the boundary between two media with different wave velocities. Each point in the wave front that touches the medium of refraction acts like a source of wavelets that propagate into that medium. These waves have the same

Equation 6·15

wave speed in medium of incidence v_i
wave speed in medium of refraction v_r
wavelength in medium of incidence λ_i
wavelength in medium of refraction λ_r

$$v_i = \lambda_i f$$

$$v_r = \lambda_r f$$

frequency as their source, and therefore the same frequency as the wave in the medium of incidence. The wave in the medium of refraction, however, where the speed is different, has an altered wavelength, because wavelength, frequency, and speed are related by $v = \lambda f$ (Eq. 6.15). The ratio of the wavelengths in the two media is equal to the ratio of the wave speeds, since these two properties of the wave are directly proportional as long as the frequency remains the same (Eq. 6.16). Thus, the change in medium results in a changed wavelength.

Construction of the refracted wave front. The procedure for finding the law of refraction is very similar to that used in the preceding section to find the law of reflection. A straight wave is incident on the refracting boundary obliquely from the left. Between each of the four successive instants shown in Figs. 6.30 and 6.31, the wave advances by 1 wavelength. The Huygens' sources on the boundary generate wavelets that propagate into the second medium with the wave speed and therefore the wavelength appropriate to that medium. The case of reduced wave speed and wavelength is illustrated in Fig. 6.30, while the case of increased wave speed and wavelength is illustrated in Fig. 6.31. In both cases the wavelets originating in points A, B, and C (and intermediate points on the boundary) have a common tangent that is the refracted wave front.

Equation 6.17

$$\frac{\text{sine}\,\theta_\text{i}}{\text{sine}\,\theta_\text{r}} = \frac{\lambda_i}{\lambda_r} \quad (a)$$

$$\frac{\text{sine}\,\theta_\text{i}}{\text{sine}\,\theta_\text{r}} = \frac{v_i}{v_r} \quad (b)$$

Law of refraction. To relate the angles of incidence and refraction, the directions of propagation have to be taken into account. This is done for both cases above in Fig. 6.32, where only one incident and one refracted wave crest from the previous figures are included. The conclusion from the application of Huygens' Principle is that the sines of the angles of incidence and refraction have the same ratio as the wavelengths (Eq. 6.17a) and, therefore, the same ratio as the wave speeds in the two media (Eq. 6.17b). This result is similar in form to Snell's Law of Refraction: (n_i sine $\theta_i = n_r$ sine θ_r , Eq 5.2, Section 5.2), a key assumption in Newton's ray model of light. We shall study this further below in Section 7.2, where we will compare and evaluate the ray and wave models in some detail.

If you look at the propagation direction of the refracted waves in Figs. 6.30 and 6.31, you will recognize that the effect of crossing the boundary can be described as follows. In the medium with the slower wave, the propagation direction is farther away from the boundary surface; in the medium with the faster wave, the propagation direction is closer to the boundary surface. You may use the tables of the sine functions (Appendix, Table A.7) to solve problems on the refraction of waves.

Reflection at the boundary. The application of Huygens' Principle to the boundary between the two media leads to reflected wavelets as well as refracted ones. One reflected wavelet is indicated in Fig. 6.30b and one is indicated in Fig. 6.31b. Since these wavelets are in the medium of incidence, their speed and wavelength are appropriate to that medium. By pursuing their formation further, we could have obtained the same sequence of diagrams as are shown in Fig. 6.27. The wavelets would interfere constructively to form a reflected wave according to the law of reflection (Eq. 6.14). Thus wave theory suggests that we should also look for reflected waves, and, in fact, by looking carefully, you can indeed identify reflected waves in the deeper water of Fig. 6.29! In other

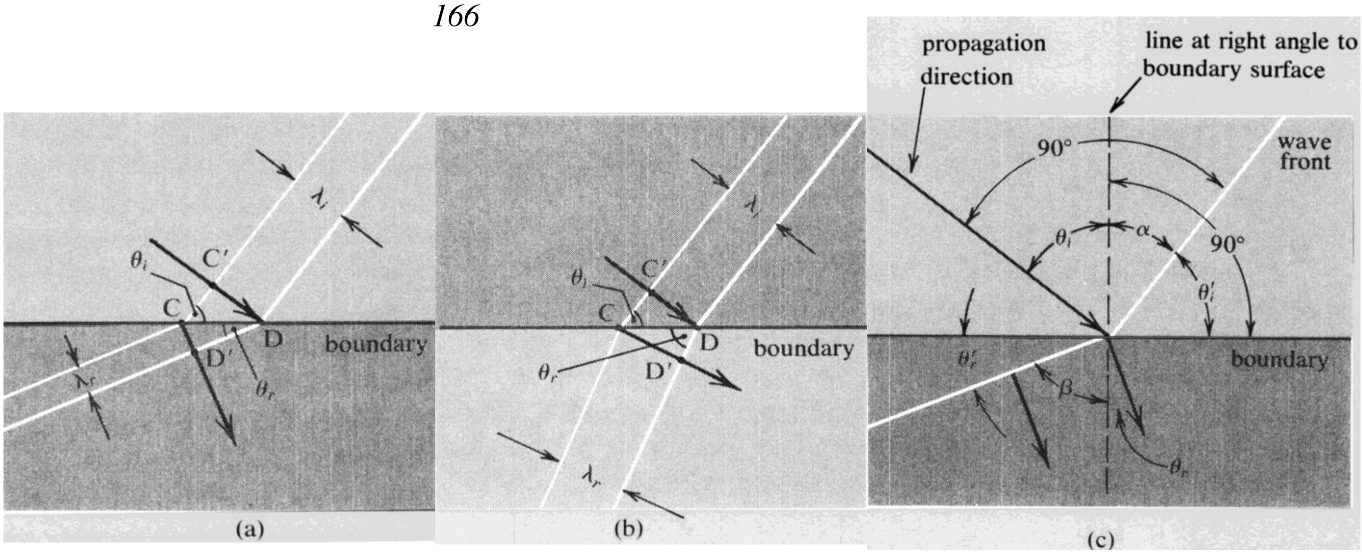


Figure 6.32 Construction of a mathematical model for wave refraction, based on Figs. 6.30 and 6.31. Note that in (a) the medium with the longer wavelength (faster speed) is at top in lighter shading. However, in (b) the medium with the longer wavelength (faster speed) is at bottom, also in lighter shading.
In both (a) and (b), Arrows C'D and CD' represent, respectively, the incident and refracted propagation directions. They are at right angles to the corresponding wave fronts (white lines) CC' and DD'. ASSUMPTION: Angle C'CD is equal to the angle of incidence θ_i and D'DC is equal to the angle of refraction θ_r ; we will prove this assumption in (c) below. The definition of the sine functions can be applied to right triangles CDC' and CDD' with the following results:

$$\text{sine } \theta_{\text{i}} = \frac{C'D}{CD} = \frac{\lambda_{\text{i}}}{CD} \quad (1)$$

$$\text{sine } \theta_{\text{r}} = \frac{D'C}{CD} = \frac{\lambda_{\text{r}}}{CD} \quad (2)$$

Divide Eq. (1) by Eq. (2)

$$\frac{\text{sine } \theta_{\text{i}}}{\text{sine } \theta_r} = \frac{\lambda_{\text{i}}}{\lambda_{\text{r}}} \qquad (3)$$

(c) Proof of ASSUMPTION asserted above: the angle of incidence θ_i equals the angle between the boundary and the wave front θ_i'. Two overlapping right angles in the medium of incidence are indicated in the figure (c) above. Both angles θ_i and θ_i' are complementary to the angle α. Consequently the two angles are equal, $\theta_i = \theta_i'$. The same construction with respect to the angle β in the second medium leads to the conclusion that $\theta_r = \theta_r'$.

words, the incident wave appears to be split by the boundary into a reflected wave and a refracted wave. Huygens' Principle has indeed served us well; however, it does not reveal how the energy carried by a wave is divided between reflection and refraction.

The occurrence of partial reflection gives an important clue about the sense of the direction of propagation of waves. By examining only the incident and refracted waves, such as in Fig. 6.29, you would not be able to determine whether the waves were incident as described at the beginning of this section (from upper left), or whether the waves were incident from the lower right and passed from the shallow water to the deeper water. The observation of reflected waves in the upper part of the photograph is evidence that the waves were incident from above.

Summary

Equation 6.3b (wave speed)

$$v = \lambda f$$

The concept of waves has its roots in water waves. More generally, waves are oscillatory displacements of a medium from its equilibrium state. Two important forms that such disturbances can take are the wave train, in which the displacement pattern repeats over and over, and the wave pulse, in which the displacements are localized in space and time. The wavelength, wave number, period, frequency, amplitude, and speed of the waves can be defined for a wave train, but only the last two of these can be defined for a pulse. The frequency, wavelength, and speed of a wave train are related by $v = \lambda f$ (Eq. 6.3b).

The wave theory is built upon the above ideas and applies to a wide variety of types of waves. The goal of wave theory is the construction of mathematical models to describe the behavior and propagation of waves. Wave theory explains and clarifies a large variety of phenomena. Such phenomena include sound, music, water waves, radio, light, constitution of the atom, traffic flow, and earthquakes.

The wave theory rests on two key assumptions about waves: 1) the superposition principle and 2) Huygen's Principle. The theoretical deductions from these assumptions can be compared with observation to identify the successes and the limitations of the wave theory.

According to the superposition principle, the displacement of the combination of two or more waves passing through the same point in space at the same time is the sum of the displacements of the separate waves. The result is constructive or destructive interference, depending on whether the separate waves reinforce or oppose one another.

Huygens' principle is used to investigate the propagation of waves. Each point in a wave front is considered as a source of circular outgoing wavelets. The amplitude and frequency of the wavelets are determined by the amplitude and frequency of the wave at the source point. The wavelets interfere constructively along their common tangent line, which is therefore the front of the propagating wave. Elsewhere, the wavelets interfere destructively and are not separately observable.

Huygens' Principle allows us to conduct thought experiments on the propagation of waves and furnishes a procedure for determining the results. We have used Huygens' Principle to understand diffraction, reflection, and refraction of waves.

The wave theory does not attempt to relate the wave speed, amplitude, and energy to properties of the medium, the wave source, and the wave absorber. These matters require more detailed working models for the three systems; their treatment is beyond the scope of this text.

List of new terms

medium (for wave propagation)
wave train
wave pulse
amplitude
frequency (f)
wavelength (λ)
wave number (k)
period ($\mathcal{T}$)
wave speed (v)

superposition
interference
constructive interference
destructive interference
node
tuned system
beats
wave packet
uncertainty principle

Huygens' Principle
wave front
propagation direction
Huygens' wavelets
diffraction
diffraction grating
reflection of waves
refraction of waves
standing waves

List of symbols

k	wave number
λ	wavelength
f	frequency
$\mathcal{T}$	period
N	number of waves
Δs	pulse width
Δt	time for one beat
Δf	frequency range $(f_1 - f_2)$
Δk	wave number range $(k_1 - k_2)$
v	wave speed
θ	diffraction angle
θ_i	angle of incidence
θ_R	angle of reflection
θ_r	angle of refraction

Problems

Here are some suggestions for problems that have to do with water waves. Observations on a natural body of water are made most effectively from a bridge or pier overhanging the water. You may observe wind-generated wave trains or pulses generated by a stone. By dipping your toe rhythmically into the water, you may be able to generate a circular wave train.

Experiments can be conducted in a bathtub or sink if natural bodies of water are not available. A pencil or comb dipped horizontally into the tub near one end can generate straight wave pulses. Dipping your finger, a pencil or a comb vertically will generate circular waves. To observe the waves, place a lamp with one shaded bulb over the bathtub so as to direct the light at the water surface and not into your eyes. You should also avoid looking at the reflected image of the bulb. Under these conditions, waves cast easily visible shadows on the bottom of the tub or on the ceiling. **Caution: You must be careful when using electricity near the bath or sink; an electrical shock from household**

current can be dangerous. Keep water away from the lamp and do not under any circumstances touch the lamp with wet hands nor while any other part of your body is wet or touching something wet.

1. Measure the speed of water waves by measuring how long they take to traverse a given distance. Describe the conditions of your observations, especially the depth of the water and the amplitude of the waves. If you observe wave trains, determine their frequency and wavelength and test Eq. 6.3b.

2. Identify the interaction(s) that are involved in the propagation of waves on a water surface.

3. Observe waves at the seashore and report qualitatively about as many of the following as you can observe.
   (a) Differences in speed of various waves.
   (b) Differences in direction of propagation.
   (c) Applicability of the superposition principle.
   (d) Effect of the depth of the water on the wave motion.
   (e) Reflection of wave fronts.
   (f) Refraction of wave fronts.
   (g) Diffraction of waves.
   (h) Transfer of energy from the waves to other systems.

4. Sand ripples are frequently observed on the ocean or lake bottom in shallow water. They are formed by the interaction of sand and water just as water waves are formed by the interaction of water and wind. Measure the wavelength of sand ripples that you observe. Comment on their propagation speed.

5. Observe reflection of water waves in your sink or bathtub. Estimate the angles of incidence and reflection as well as you can and compare your results with the law of reflection for waves.

6. Observe single-slit diffraction of water waves in your sink or bathtub. Report the slit width you found most suitable and other conditions that helped you to make the observations.

7. Several different diffraction gratings diffract water waves with a wavelength of 0.03 meter. Find the diffraction angle for a diffraction grating with a slit spacing of (a) 0.30 meter; (b) 0.10 meter; (c) 0.05 meter; (d) 0.025 meter.

8. Water waves are diffracted by a grating with a slit spacing of 0.30 meter. Find the wavelengths for the waves when the diffraction angle is (a) 10°; (b) 25°; (c) 60°.

9. Find the result of superposing the following three waves:
   wave A -- wavelength (λ) = 6 centimeters (cm), amplitude = 3 cm;
   wave B -- λ = 3 cm, amplitude = 2 cm;
   wave C -- λ = 2 cm, amplitude = 1 cm.
   Start from a point where all three waves interfere constructively; keep plotting until all three waves again interfere constructively.

10. Sound waves in air have a wave speed of 340 meters per second. Find the wavelength and wave number of the following sound waves: (a) middle C, frequency (f) = 256 per second; (b) middle A, f = 440 per second (c) high C, f = 1024 per second.

11. Use the paper strip analogue (Fig. 6.24) to study diffraction of waves. Report the wavelength, "slit" separation, and diffraction angle(s) for three different "gratings." Choose λ/d small (0.5), medium (2.0), and close to one for the three cases. (Note: one grating may give several diffraction angles, according to whether the waves from adjacent slits are 1, 2, 3, ... wavelengths out of step.) Make as many paper strips as you feel necessary to help you.

12. Use the paper strip analogue (Fig. 6.24) to study diffraction from only two slits. Measure and/or use geometrical reasoning to find the angles of diffraction amplitude maxima (constructive interference) and diffraction amplitude minima (destructive interference). Compare your results with those obtained for a diffraction grating and describe qualitatively the reasons for similarities and differences.

13. Identify one or more explanations or discussions in this chapter that you find inadequate. Describe the general reasons for your judgment (conclusions contradict your ideas, steps in the reasoning have been omitted, words or phrases are meaningless, equations are hard to follow, . . .), and make your criticism as specific as you can.

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