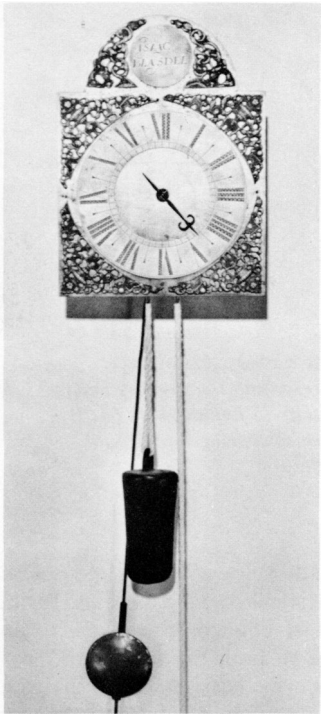
Chapter 15: Periodic Motion



Have you ever wondered how extensively human civilization depends on clocks? Western culture is especially time conscious, since daily schedules in the complex worlds of business, industry, and education require synchronized cooperation by many individuals. Sometimes it seems as though clocks are the masters and humanity is enslaved!

Clocks are instruments that make possible the operational definition of time intervals. Each clock has a regulating mechanism, such as a pendulum, a balance wheel, or a vibrating crystal, which moves in a repeating pattern (called *periodic motion)* and defines equal time intervals, one after another. In this chapter, we will formulate a mathematical model for periodic motion and apply it to the solar system, the pendulum, and the elastic oscillator. We will not be concerned with other aspects of clocks, such as their energy sources, the internal connections that regulate the movement, and the face or dial on which the elapsed time is indicated.

Periodic motion also occurs under natural conditions in the macro domain. The child in her swing, the bird swaying gently on a tree branch, the bicycle wheel spinning on its axle, your arm swinging by your side as you walk—all these are examples of periodic motion from the everyday world.

There are several sections in this text where we have already referred to periodic motion. In Section 1.5, the operational definition of time intervals was based on periodic motions such as the earth's revolution on its axis and the swinging of a pendulum. In Section 3.4, we introduced the inertial balance, whose periodic motion is the basis for the operational definition of inertial mass. In Section 6.1, we introduced the oscillator model for a medium that permits wave propagation, and we described the periodic motion of each oscillator (Fig. 6.4).

15.1 Properties of periodic motion

Periodic motion can occur in a system only when the objects in the system interact with one another in such a way that they remain near one another. In the absence of interaction, the objects would move apart and not return to repeat their motion once they had passed each other. Even though systems carrying out periodic motion will eventually stop moving, they do have a stable existence over many cycles of their motion, as shown by the example of the solar system.

In this chapter we will examine periodic motion from the viewpoint of the Newtonian theory. The two principal quantities that are used to describe periodic motion of a particle are the time required for one cycle, which is called the *period* ($\mathcal{T}$, seconds), and the radius or width of the orbit (R, meters). We will construct a mathematical model for periodic motion and thereby relate the period and radius to the mass of the particle and the force that maintains the periodic motion.

The qualitative nature of the relationship among period, radius, mass, and force is easy to infer. The moving particle has a velocity

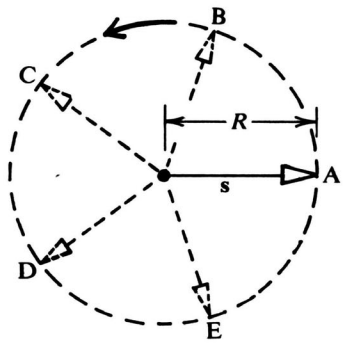


Figure 15.1 Circular motion with radius R and period T. In the time T, the arrow representing the relative position of the orbiting particle rotates once around the circle counterclockwise.

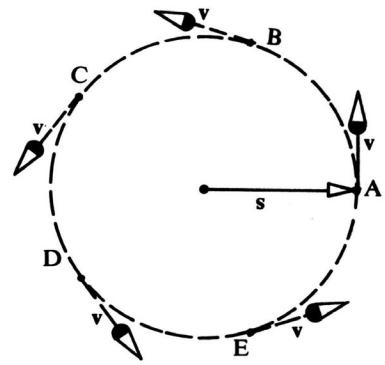


Figure 15.2 Arrows representing the instantaneous velocity at five instants (A, B, C, D, E) during one orbital revolution.

whose magnitude depends on the radius (distance traveled) and period (time required) of the orbit. Since the particle is moving back and forth, the velocity cannot be constant but must change, giving rise to an acceleration. The acceleration is related to both the net force and inertial mass according to Newton's second law (Eq. 14.6, $\mathbf{F}_{net} = M_I\mathbf{a}$). As explained in Section 14.4, we can assume that the inertial mass of a particle is equal to its gravitational mass. Therefore, from now on we will refer to the mass without specifying it further; thus, $\mathbf{a} = \mathbf{F}_{net}/M$.

15.2 A mathematical model for circular motion

Circular motion with a constant speed is a particularly simple example of periodic motion. The moon in its orbit around the earth, the tetherball whirling at the end of its string, and the child on the merry-go-round exemplify more or less closely this kind of periodic motion. The arrow representing the position of the moving particle in a diagram rotates around the circle (Fig. 15.1). The acceleration in this example arises only from changes of direction of the instantaneous velocity, since the instantaneous speed (magnitude of the velocity) does not change (Section 13.2). To apply Newton's theory, we must first find how the acceleration is related to the radius and period.

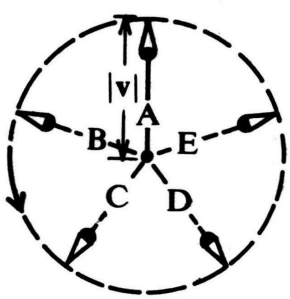


Figure 15.3 Diagram of the velocities at five instants during one revolution of the orbiting particle. Since the magnitude of the velocity is constant, the velocity arrows all extend from the origin to the dashed circle of radius |v|.

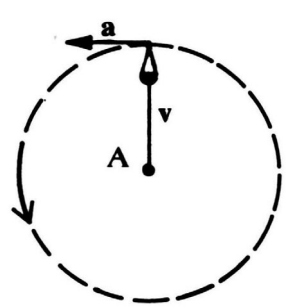


Figure 15.4 The arrow representing the acceleration is in the direction of the velocity change, which is to the left as the velocity arrow rotates around the circle counterclockwise.

Equation 15.1

instantaneous velocity = **v**
instantaneous speed = v
average speed = v_{av}

$$|\mathbf{v}| = v = v_{av}$$

Equation 15.2 (velocity of particle moving in a circle)

distance traveled = Δs
time elapsed = Δt
orbital radius = R
orbital period = $\mathcal{T}$

$$|\mathbf{v}| = v_{av} = \frac{\Delta s}{\Delta t} = \frac{2\pi R}{\mathcal{T}}$$

From here on we will merely write "speed" and "velocity"for the instantaneous quantities. The average quantities will be designated as such so you may identify them properly.

Equation 15.3

acceleration = **a**

$$|\mathbf{a}| = \frac{2\pi|\mathbf{v}|}{\mathcal{T}}$$

Equation 15.4 (Acceleration of particle moving in a circle)

$$|\boldsymbol{a}| = 2\pi\left(\frac{2\pi R/\mathcal{T}}{\mathcal{T}}\right)$$

$$= \frac{4\pi^2 R}{\mathcal{T}^2}$$

Velocity of circular motion. The instantaneous velocity is directed along the circumference of the circular orbit, at right angles to the position arrow (Fig. 15.2). The magnitude of the instantaneous velocity is equal to the instantaneous speed, and this, in turn, is equal to the average speed since the latter does not vary (Eq. 15.1). The average speed is equal to the circumference of the circular orbit divided by the period (Eq. 15.2, $|v| = 2\pi R/T$).

It is instructive to make a diagram in which the instantaneous velocity arrows are compared (Fig. 15.3). The tails of all the arrows are placed at the origin of the coordinate frame and the heads of the arrows fall on the circle whose radius is the velocity magnitude given by Eqs. 15.1 and 15.2. You can see that the velocity diagram (Fig. 15.3) is very similar to the position diagram (Fig. 15.1). One way of describing the circular motion is to say that both the position arrow and the velocity arrow rotate once around their circles during one period.

Acceleration in circular motion. The easiest way to infer the direction and magnitude of the acceleration is to reason from the formal similarity of the relations "position-velocity" and "velocity-acceleration." We have just pointed out that the geometry of the position arrow, which rotates counterclockwise around a circle (radius *R)* in one period (Fig. 15.1), is analogous to the geometry of the velocity arrow, which also rotates counterclockwise around a circle (with radius $|v|$) in one period (Fig. 15.3). The magnitude of the acceleration is therefore given by a formula like Eq. 15.2, except using the radius of the velocity circle instead of the radius of the position circle (Eq. 15.3). In fact, the head of the velocity arrow travels around the circumference of a circle, a distance of $2\pi v$; therefore, over the full circle $a = \Delta v/\Delta t = 2\pi v/\mathcal{T}$. If you replace v with its value from Eq. 15.2 you find Eq. 15.4, in which the magnitude of the acceleration is directly proportional to the orbital radius and inversely proportional to the period raised to the second power.

The direction of the acceleration can be found by using the analogy of the position circle in Fig. 15.2 to the velocity circle in Fig. 15.3. The velocity arrow in Fig. 15.2 points at right angles to the position arrow, and to the left. The acceleration implied by Fig. 15.3 is therefore directed at right angles to the velocity, and to the left (Fig. 15.4). The position, velocity, and acceleration at instant A are summarized in Fig. 15.5. You can see that the acceleration is directed from the

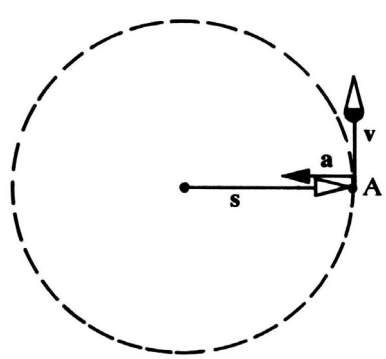


Figure 15.5 Comparison of the position, velocity, and acceleration of a particle moving at uniform speed along a circular path. The tails of the velocity and acceleration arrows are placed at the position of the particle (A). The velocity points along the tangent to the path; the acceleration points toward the center of the circle.

point A toward the *center* of the circle, a result that also applies at any point on the circle (Fig. 15.3, B, C, . . .) . In his studies of circular planetary orbits, Newton therefore introduced the term *centripetal acceleration* for the acceleration of any object moving in a circle at constant speed.

Equation 15.5 (Centri - petal force for motion in a circle)

force (newtons) = **F**
mass (kg) = M

$$|\mathbf{F}_{net}| = M|\mathbf{a}|$$
(Newton's Second Law)

$$|\mathbf{F}_{net}| = \frac{4\pi^2 MR}{\mathcal{T}^2}$$
(from Eq. 15.4)

Centripetal force. When you use Newton's second law ($\mathbf{F}_{net} = M\mathbf{a}$) to calculate the force required to cause this acceleration (Eq. 15.5), you find that the magnitude of the force is constant and that it is always directed toward the center of the circle. The force is therefore often called a *centripetal force.* The magnitude of the force is directly proportional to the mass of the particle and the radius of the circle; it is inversely proportional to the second power of the orbital period (Eq. 15.5). This equation is a mathematical model for the centripetal force during circular motion. The application of this model to a simple laboratory experiment is described in Fig. 15.6. In the experiment, a weight hanging from a string supplies the centripetal force, which is transmitted by the string to the orbiting weight.

"A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as a centre.... Of this sort is gravity, by which bodies tend to the centre of the earth; magnetism, by which iron tends to the lodestone; and that force, whatever it is, by which the planets are continually drawn aside from the rectilinear (straight line) motions, which otherwise they would pursue, and made to revolve in curvilinear orbits."

Isaac Newton
Principia, 1687

15.3 The solar system and gravitation

Newton applied the theory of circular motion and its extension for elliptical motion to the moon and the planets in the solar system. He used the observations of orbital motion to deduce the centripetal forces that were acting. He then identified these forces with the terrestrial force of gravity and thereby unified terrestrial and celestial phenomena.

Newton took the point of view that the sun exerted the force that kept the planets in their orbits. Unlike his predecessors Copernicus, Kepler, and Galileo, Newton had a mathematical model relating force to motion ($\mathbf{F}_{net}\Delta t = \Delta\mathcal{M}$, or $\mathbf{F}_{net} = M\mathbf{a}$). This was a powerful method for testing whether the heliocentric point of view resulted in a simple explanation for the observations that had been made on the solar system. In particular, Newton invented a simple mathematical model for the force exerted

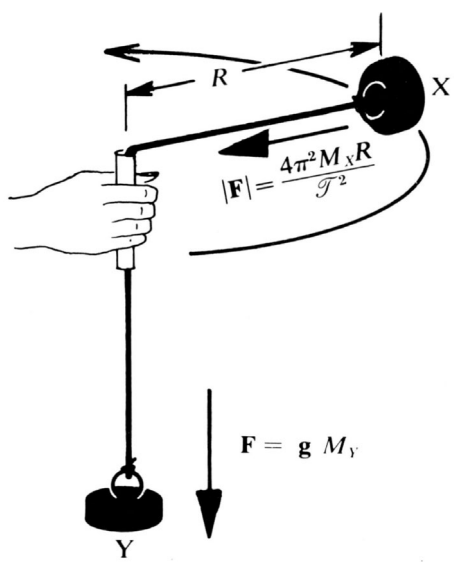


Figure 15.6 Weight X is twirled in a circle to hold weight Y in mechanical equilibrium. The centripetal force on weight X is equal in magnitude to the gravitational force $|g|M_Y$ acting on weight Y. Measurements of the radius R, the period T, and the masses of the two weights lead to a direct experimental test of Eq. 15-5.

". . . the natural motion of the earth as a whole, like that of its parts, is towards the centre of the Universe; that is the reason why it is now lying at the centre . . . light bodies like fire, whose motion is contrary to that of the heavy, move to the extremity of the region which surrounds the centre."
Aristotle, On the Heavens, 4th century B.C.

Claudius Ptolemy (approx. 140 A.D.) was probably an Egyptian Greek who lived for a period in or near Alexandria in Egypt (127-ca. 150 A.D.). His two great works, the Almagest (on mathematical astronomy) and the Geography (on mathematical geography) remained the standard text references in their respective fields for over 14 centuries.

Nicolaus Copernicus (1473-1543), a Polish astronomer, proposed replacing the complex geocentric universe with a simpler, sun-centered (heliocentric) system. Unfortunately, to advance such views in the early 16th century was heresy, and even Martin Luther spoke of Copernicus as "the fool who would overturn the whole science of astronomy." While the heliocentric system was simpler, the existing observations did not decisively favor either system, and there was not yet a consensus among scientists. Consequently, Copernicus would not permit his book on the subject to be published until he lay on his deathbed.

by the sun on the planets. This model relates the magnitude of the force acting on each planet to the distance between the sun and the planet.

Motion in the solar system. Astronomy, the oldest science, has been important for millenia, because it has enabled us to anticipate seasonal changes and to schedule agricultural activities. The observed motion of the sun, moon and stars around the earth was first attributed to gods and goddesses. Later, Greek philosophers explained the motion by means of celestial concentric rotating spheres to which the heavenly bodies were attached. The earth was at the center of all the spheres; hence, this model is called *geocentric* (earth-centered). This model seems intuitively reasonable, and it became generally accepted.

However, the motion of the sun and the planets in the sky was more complicated than that of the stars. In the third century BC, a Greek astronomer, Aristarchus of Samos, suggested that *"the fixed stars and the sun remain unmoved, that the earth revolves about the sun on the circumference of a circle, the sun lying in the middle of the orbit"* (as reported by Archimedes in *The Sandreckoner).* We will call this model *heliocentric* (sun is *helios* in Greek, thus sun-centered). This model attracted little attention throughout ancient and medieval times.

Ptolemy's geocentric theory. The major trend of ancient thought elaborated the geocentric point of view, which offered three great advantages: first, philosophical doctrines required that the earth be stationary at the center of the universe; second, the spherical motion of the heavenly spheres was a "natural" and perfect motion; third, the data gathered by observers on earth could be used for the prediction of stellar movements in a geocentric reference frame but were not sufficiently accurate to permit their correct transformation to a heliocentric reference frame. Claudius Ptolemy developed this theory to a perfection that served mankind until it was abandoned in favor of the heliocentric theory after more than fourteen hundred years.

The Copernican heliocentric theory. One of the weaknesses of the Ptolemaic theory was the need to make it more complicated as astronomical data improved in accuracy. In fact, Ptolemy himself recognized that not all celestial spheres moved around the earth as center of rotation. In the sixteenth century, Nicolaus Copernicus revived the heliocentric picture of the universe and thereby touched off a controversy between religious dogma and science that lasted for a hundred years. Copernicus realized that the rotation of the earth on its axis could explain the motion of the fixed stars, which he placed on an immobile celestial sphere. In his theory, planets were on smaller concentric spheres, with the sun at the center of the universe. Unlike earlier proponents of the heliocentric point of view, however, Copernicus had the data with which he could find the period of motion and the orbital radius of each planet around the sun. His unit of distance was the distance between the sun and the earth, known as the astronomical unit (AU). Copernicus' values for the planets' periods and orbital radii were very close to the modern values (Table 15.1).

Copernicus succeeded in showing that heliocentric theory was as

TABLE 15.1 DATA ON THE SOLAR SYSTEM

Planet	Radius (m)	Mean radius of orbit (m)	(AU*)	Period (years)	$(\text{Radius})^3$ $(\text{AU*})^3$	$(\text{Period})^2$ $(\text{years})^2$
Mercury	2.5×10^6	5.8×10^{10}	0.39	0.24	0.059	0.058
Venus	6.1×10^6	1.1×10^{11}	0.72	0.62	0.38	0.38
Earth	6.4×10^6	1.5×10^{11}	1.00	1.00	1.00	1.00
Mars	3.4×10^6	2.4×10^{11}	1.52	1.9	3.5	3.6
Jupiter	7.2×10^7	7.8×10^{11}	5.2	11.9	140.	141.
Saturn	5.8×10^7	1.4×10^{11}	9.6	30.	890.	900.
Uranus**	2.7×10^7	2.9×10^{11}	19.2	84.	7100.	7000.

*AU stands for astronomical unit, a distance measure equal to the mean orbital radius of the earth (1 AU = 1.5×10^{11} m).
**Not known in Copernicus' and Newton's times.

Johannes Kepler (1571-1630) was born in Weil der Stadt, Germany, and studied mathematics at the University of Tübingen. After obtaining his degree, Kepler became Tycho Brahe's assistant at Prague and eventually succeeded him as Court Mathematician. Kepler argued for the Copernican system in his first book The Cosmographic Mystery *(1597). Through his studies of the orbit of Mars, Kepler arrived at his first two laws, and he published his findings in the* New Astronomy or Celestial Physics *(1609). The third of his great laws was announced in* The Harmonies of the World *(1619). In all of his research, Kepler benefited immensely from the huge collection of accurate, long-term astronomical measurements that Tycho bequeathed to him.*

good as geocentric theory in summarizing the astronomical data. His theory, however, did not have a decisive scientific advantage; it merely appealed to a different sense of order or simplicity than did the geocentric theory. It is now recognized that Copernicus started a scientific revolution by advancing a new interpretation of data, but that others, especially Isaac Newton, really exploited the new point of view.

Kepler's laws. New and especially accurate data on the motion of Mars were collected by the Danish astronomer Tycho Brahe (1546-1601) and were left for his assistant Johannes Kepler to analyze. After painstaking work, Kepler concluded that Mars did not carry out uniform circular motion, either geocentric or heliocentric. He therefore used the observations of Brahe, as a surveyor would use his sightings, to identify the geometrical shapes of the orbits of the earth (the surveyor's "base") and of Mars relative to the sun as fixed point.

Kepler's conclusions are summarized in three laws that bear his name. The first law states that the planets' orbits have the shape of ellipses, with the sun located at one focus of the ellipse (Fig. 15.7). Actually, the ellipses of most planets are quite close to circles, in that the two diameters differ in length by only a few percent. This fact explains in part why the model of circular orbits had not been discarded earlier. Kepler's second law points out that a planet does not move with the same speed along all parts of its orbit. It moves faster than average when it is close to the sun (point A, Fig. 15.7) and slower when it is far from the sun (point B, Fig. 15.7), in such a way that the line connecting it to the sun sweeps out equal areas in equal times. Here also the deviations from uniform motion are so small that they were not detected before Brahe's accurate observations.

Equation 15.6
(Kepler's Third Law)

planetary period (years) $= \mathcal{T}$
orbital radius (AU) = R

$$\mathcal{T}^2 = R^3 \qquad (a)$$

planetary period (sec) $= \mathcal{T}$
orbital radius (m) = R

$$\mathcal{T}^2 = (3\times10^{-18})R^3 \quad (b)$$

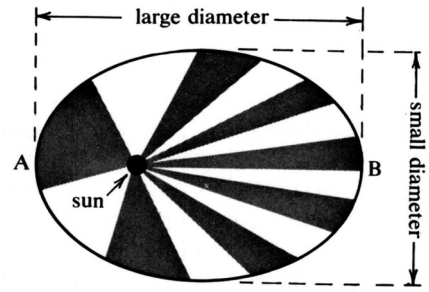


Figure 15.7 Kepler's Laws
***Kepler's First Law**: The orbit of a planet around the sun is in the shape of an ellipse with the sun at one focus.*

***Kepler's Second Law**: the line from the sun to the planet sweeps out equal areas in equal times.*

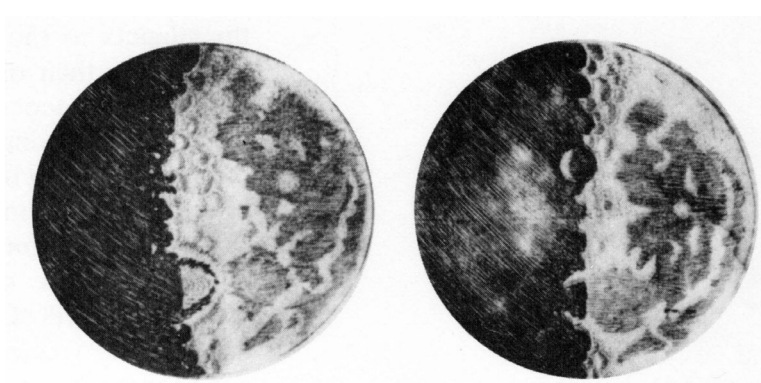
Kepler's first two laws apply to the motion of each planet separately. Kepler's third law relates the motions of all the planets together. In modern terminology, the third law states that the planetary period (in years) to the second power equals the mean orbital radius (in astronomical units) to the third power (Eq. 15.6a). Note how the orbital motion of the earth furnishes the units of distance and time. Note also that Kepler's third law, though published a whole decade after his first two laws, made use basically of the data of Copernicus and did not depend on Kepler's other discoveries. For later use, we state Kepler's third law in ordinary units (meters, seconds) in Eq. 15.6b.

The nature of the sun and planets. One of the important assumptions of geocentric theory was that terrestrial matter was essentially different from the celestial. The latter was perfect, uniform, and spherical in shape and motion; the former was irregular, rough, and massive, falling toward the ground when unsupported. Copernicus had addressed himself to a description of the relative motion of the heavenly bodies without concerning himself about its causes. Kepler's search for simple mathematical patterns in the description had been motivated by his vision that a single force exerted by the sun was responsible. However, he had no quantitative techniques for analyzing this problem, but he speculated that magnetism might be the agency. In his view, the planets and the sun were magnets.

The first direct observations of heavenly bodies to give evidence of their nature were made by Galileo with a telescope that had been recently invented and that he had improved. Galileo examined the moon, the sun, the Milky Way, the planet Jupiter, and the planet Venus.

What Galileo saw supported his belief in the heliocentric Copernican

Figure 15.8 (below) Two drawings made by Galileo from his telescopic observations of the moon. These drawings showed the mountains and valleys on the moon and directly contradicted existing beliefs. He published them in his "best seller" Sidereus Nuncius (The Starry Messenger, 1610). *Galileo's extraordinary observations, published as a 63-page pamphlet in popular Italian rather than the traditional Latin, provided clear-cut support for the heliocentric system over the then widely-accepted geocentric theory. Unfortunately, the geocentric system had been incorporated into Catholic dogma, and at the same time as Galileo was advocating the heliocentric system, the Church was becoming more conservative as Protestantism grew. Galileo's conflict with the Church finally led to his "trial" by the Inquisition, the threat of torture, and his public "confession" to the crime of heresy! This was, and remains, an unforgettable episode, a searing reminder of the importance of the struggle required to develop civil society with an independent justice system and limits on religious authority.*



Galileo Galilei (1564-1642) was born in the year of Shakespeare's birth and Michelangelo's death. By the age of 25, he was a teaching member of the University of Pisa. Galileo's intense criticism of Aristotelian natural philosophy provoked a controversy that drove him to Padua in 1592, where he constructed his first telescope and began his battle on behalf of the heretical Copernican theory. His fame became so widespread that he was even recalled to Pisa as Court Mathematician and Philosopher. In spite of prohibitions from the Church, Galileo published his Dialogue Concerning the Two Chief Systems of the World*, which praised Copernican theory at the expense of orthodox Ptolemaic theory. Galileo was again brought before the Inquisition. Weakened by age and fearing for his safety, he publicly abjured his belief in the heliocentric system and pledged not to discuss nor write about it again. However, Galileo survived to publish his masterpiece on motion on earth*: Discourses Concerning Two New Sciences (1638).

Galileo's extraordinary example of the value of individual freedom in scientific inquiry, and the high price he paid for it, stimulated the search for a less destructive relationship between science and religion. Today, we sometimes take freedom of inquiry and freedom of religion for granted, but this was (and is) not always so. Maintaining the uneasy balance between reason and faith (or freedom and security) requires ongoing vigilance and hard work.

model of the solar system, for many reasons: 1. The moon exhibited a rough landscape with mountains and valleys much like the earth and was not perfect, spherical, and uniform (Fig. 15.8). 2. The sun's disk likewise was not uniformly bright, but showed dark "spots" that moved on its surface; this further contradicted the idea of celestial perfection and strongly suggested that the sun itself was rotating. 3. The Milky Way was not a continuous ribbon of light but was made up of many faint stars, individually invisible, that could not conceivably serve the purpose of providing light for men to see at night. 4. The planet Jupiter was surrounded by four small bright stars that, Galileo showed, were really satellites revolving around Jupiter much like the moon revolves around the earth. This was direct observation of orbital motion not centered on the earth. 5. Venus exhibited moon-like phases; thus Venus must *reflect* sunlight like the moon and was not self-luminous, again contradicting Aristotle's assertions about celestial objects. 6. Finally, Venus showed great variations in apparent size and hence in distance from the earth; therefore, Venus *cannot* move in a circular orbit around the earth, as required by the geocentric system. On the other hand, as Galileo enthusiastically explained, the Copernican system could easily explain every detail of his observations of Venus. In the last hundred years, spectroscopy has offered final chemical confirmation of Galileo's hypothesis that "terrestrial" and "celestial" matter shared the same fundamental nature and thus were subject to the same scientific laws.

Galileo publicized his findings widely, but he did not convince most proponents of the geocentric view, many of whom simply refused to accept the telescopic observations as evidence. Because his findings were believed to threaten the Church and the existing social order, Galileo was required to cease his teaching and even to deny the Copernican theory. Nevertheless, the process of free inquiry had begun to show significant results, and the effort to understand matter and motion both in the heavens and on earth had begun.

The Newtonian theory of the solar system. On the basis of his theory of motion, Newton investigated the shapes of particle orbits around an attracting center that subjected the particles to a centripetal force. The first result was that the lines from a particle to the attracting center swept out equal areas in equal times. Since this finding was in accord with Kepler's second law, it was clear that planetary motion was caused by a centripetal force attracting the planets to the sun and that there was no need for other forces propelling them along their orbits.

Newton's second result concerned the magnitude of the centripetal force. By following a line of reasoning similar to that in Section 15.1, Newton found a mathematical model for the centripetal acceleration of a particle moving in an elliptical orbit. The centripetal acceleration did not have a constant magnitude at all points of the orbit, but varied inversely as the second power of the particle's distance from the attracting center. Consequently the required centripetal force, proportional to the acceleration according to Newton's second law, also must vary inversely as the second power of the distance. Thus Kepler's first law regarding the shape of the orbit permitted Newton to infer

Equation 15.7

force on planet (newtons) $= F_{planet}$

mass of planet (kg) $= M_{planet}$

orbital radius (m) $= R_{planet}$

orbital period (sec) $= \mathcal{T}_{planet}$

$$\left|\mathbf{F}_{planet}\right| = \frac{4\pi^2 M_{planet} R_{planet}}{\left(\mathcal{T}_{planet}\right)^2}$$

Equation 15.8

$$\left|\mathbf{F}_{planet}\right| = \frac{4\pi^2 M_{planet} R_{planet}}{3\times10^{-18}\left(R_{planet}\right)^3}$$

$$= \frac{1.3\times10^{-19} M_{planet}}{\left(\boldsymbol{R}_{planet}\right)^2}$$

"Hitherto [I] have explained the phenomena of the heavens and of our sea by the power of gravity . . . and even . . . the remotest [motions] of the comets . . . But I have not been able to discover the cause of [the] properties of gravity from phenomena, and I frame no hypotheses; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy."

Isaac Newton
Principia, 1687

the magnitude of the force, just as Kepler's second law had permitted Newton to infer the direction.

Newton was also able to show that Kepler's third law led to the conclusion that the force holding the planets in their orbits varied inversely as the second power of the distance from the sun to the planet. For this reasoning, the planetary orbits can be described approximately as circles. Then the centripetal force acting on the planets is given by Eq. 15.7 (from Eq. 15.5) in terms of the orbital radius and period. Now, since Kepler's third law relates the period to the radius (Eq. 15.6b), reference to the orbital period can be eliminated from Eq. 15.7 to give a mathematical model for the centripetal force that depends only on the radius (Eq. 15.8). You can see that the force varies inversely as the second power of the orbital radius and directly as the mass of the planet. Kepler's first and third laws, therefore, led Newton to the same conclusion, a result that must have increased Newton's confidence in his findings.

Newton furthermore proposed a dramatic solution to the problem of the nature of the interaction. Kepler had speculated that the force was magnetic, and Descartes had worked out a "theory of vortices" in which the interaction was transmitted by a swirling fluid. But Newton argued that the interaction was gravitational, that it was of the same type as the interaction that causes an apple on earth to fall to the ground, that it varied inversely as the second power of the distance of the particle from the attracting center, and that it varied directly as the mass of the particle. The gravitational force exerted by the sun on a planet is given in Eq. 15.8, and the force exerted by other gravitating bodies (as by the planets or by their satellites) is given by a similar mathematical model, but with a different numerical factor. To justify this proposal, Newton showed that his theory could correctly account for the orbital motion of the moon as it is described in Table 15.2.

The force of gravity exerted by the earth governs the motion of the moon around the earth. The acceleration caused by this force is equal to the acceleration of gravity; at the surface of the earth, this acceleration is 10 meters per second per second (Eq. 14.14). As the force of gravity decreases with increasing distance from any gravitating body, the acceleration of gravity decreases likewise, inversely as the second power of the distance. The mathematical model in Eq. 15.9 describes this variation and gives the correct value for the acceleration of

Equation 15.9

acceleration of gravity (m/sec/sec) $= \mathbf{g}$

distance from earth's center (m) $= R$

$$|\mathbf{g}| = \frac{4.0\times10^{14}}{R^2}$$

TABLE 15.2 DATA ON EARTH SATELLITES

Satellite	Orbital radius* (m)	Orbital radius* (earth radii)	Orbital period (sec)	Orbital period (days)
Moon	3.8×10^8	60.	2.3×10^6	27.3
Syncom	4.3×10^7	6.7	8.6×10^4	1.0
Explorer 1	6.6×10^6	1.03	5.1×10^3	0.062

*The radius of the earth is a convenient unit to use for describing distances to earth satellites.

gravity at the earth's surface, at a distance of one earth radius from the earth's center (Example 15.1). The acceleration of gravity predicted by this theory at the position of the moon is much smaller, as calculated also in Example 15.1.

What is the observed centripetal acceleration of the moon? This can be calculated from the moon's orbital data and the mathematical model in Eq. 15.4. It is found to agree closely with the prediction (Example 15.2), thus lending further support to Newton's theory of gravitation as the binding force of the solar system. This was the final blow to the ancient view, according to which terrestrial and celestial phenomena were qualitatively different.

This extraordinary conceptual transformation (some would say revolution) came about as the result of two key developments: first, new mathematical techniques, including the rectangular coordinates of Descartes and methods of dealing with infinitesimal quantities invented by Newton and Leibniz (the calculus), and second, Galileo, Brahe, Kepler, Hooke, Huygens and others' use of experiments as a way to gather data and thus understand the details of real world motions. This is, indeed, an example of how a scientific breakthrough really rests squarely on the contributions of many other individuals.

EXAMPLE 15·1. **Use Eq. 15·9 to find the acceleration of gravity caused by the earth at the surface of the earth and at the position of the moon.**

(a) At the surface of the earth (Table 15·1), $R = 6.4 \times 10^6$ m

$$|\mathbf{g}| = \frac{4.0 \times 10^{14}}{R^2} = \frac{4.0 \times 10^{14}}{(6.4 \times 10^6 \text{ m})^2} = \frac{4.0 \times 10^{14}}{4.0 \times 10^{13}}$$

$$= 10 \text{ m/sec/sec} = 10 \text{ newtons/kg}$$

(b) At the position of the moon (Table 15·2), $R = 3.8 \times 10^8$ m

$$|\mathbf{g}| = \frac{4.0 \times 10^{14}}{R^2} = \frac{4.0 \times 10^{14}}{(3.8 \times 10^8 \text{ m})^2} = \frac{4.0 \times 10^{14}}{14 \times 10^{16}}$$

$$= 2.8 \times 10^{-3} \text{ newton/kg} = 2.8 \times 10^{-3} \text{ m/sec/sec}$$

EXAMPLE 15·2. **Use Eq. 15·4 to find the centripetal acceleration of the moon.**

orbital radius $R = 3.8 \times 10^8$ m, orbital period $\mathscr{T} = 2.3 \times 10^6$ sec

$$|\mathbf{a}| = \frac{4\pi^2 R}{\mathscr{T}^2} = \frac{4 \times 10 \times 3.8 \times 10^8 \text{ m}}{(2.3 \times 10^6 \text{ sec})^2} = \frac{1.5 \times 10^{10}}{5.3 \times 10^{12}}$$

$$= 2.8 \times 10^{-3} \text{ m/sec/sec}$$

Henry Cavendish (1731-1810) inherited a fortune through the death of an uncle and withdrew from society to devote himself to scientific pursuits. Unfortunately, the same shyness that produced withdrawal from society also made him reluctant to publish his manuscripts. Although Cavendish was known as a chemist, he was the first to measure gravitational forces directly. His unpublished experiments were later found (by Maxwell in 1879) to have anticipated some of the electrical discoveries of Coulomb and Faraday.

Law of universal gravitation. Laboratory experiments to test Newton's mathematical model for the gravitational force had to await the construction of the delicate apparatus that was necessary. About the

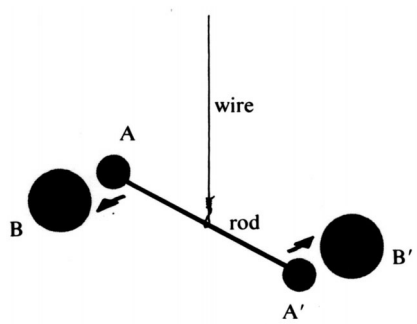


Figure 15.9 Cavendish's experiment to measure the force of gravity between lead spheres A-B and A′-B′. The top end of the wire is clamped firmly. As the rod rotates around the wire due to the gravitational attraction between the lead spheres, the bottom end of the wire is twisted elastically until it comes to mechanical equilibrium subject to the elastic and gravitational forces. The position of the rod is recorded accurately, and the spheres B and B′ are then moved to the other side of spheres A and A′. The rod then rotates and comes to equilibrium at a slightly different position. The small change in the angular position of the rod is measured by using a light beam reflected from a mirror mounted on the rod.

year 1800, Henry Cavendish succeeded by a method described theoretically by Newton and illustrated in Fig. 15.9. Cavendish found what Newton had surmised, that the gravitational force of interaction is directly proportional to the product of the masses of the two interacting bodies and varies inversely as the second power of the distance between them (Eq. 15.10). The gravitational constant has a small numerical value in the units of newtons, kilograms, and meters because the force of gravity is extremely weak unless one of the two interacting bodies has a very large mass. Equation 15.10 is called the *law of universal gravitation* because it describes the ability of all objects, terrestrial and celestial, to participate in gravitational interaction. Applications of the law of universal gravitation are described in Examples 15.4 and 15.9 at the end of this chapter.

Equation 15.10 (Law of Universal Gravitation)

force of gravity = $\mathbf{F}_G$
masses of interacting bodies = M_1, M_2
distance between centers of bodies = R
gravitational constant = G

$$|\mathbf{F}_G| = G\frac{M_1 M_2}{R^2} \qquad (a)$$

$$G = 6.7\times10^{-11}\,\frac{newton\bullet m^2}{kg^2} \qquad (b)$$

15.4 The pendulum

The simple pendulum model. A pendulum is a system that carries out periodic swinging motion in gravitational interaction with the earth (Fig. 15.10). At equilibrium, the system hangs in the vertical direction. When the system is released after being displaced from its equilibrium position, it swings back through the equilibrium position

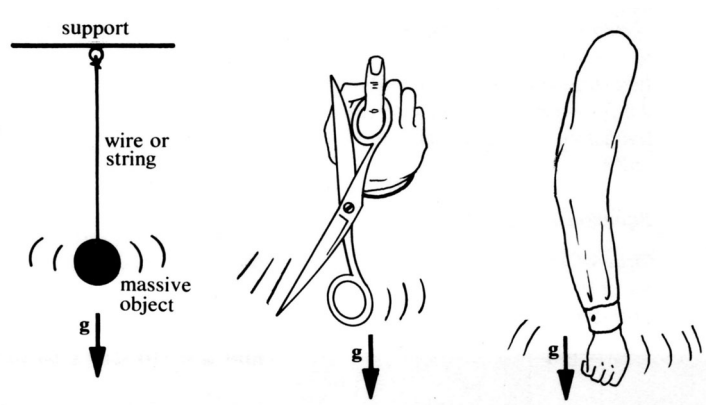


Figure 15.10 A pendulum consists of a massive object that is supported, but is free to swing in the gravitational field.

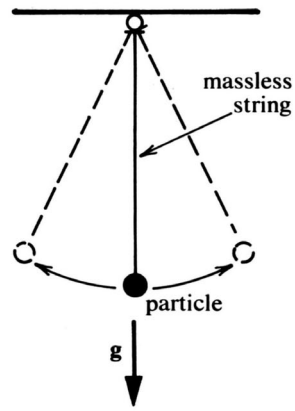


Figure 15.11 The simple pendulum is a working model for a real pendulum.

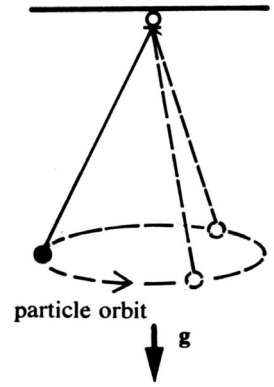


Figure 15.12 The simple pendulum particle can also swing in an orbit around the equilibrium position.

to the other side. A *simple pendulum* is a working model for a pendulum; it consists of a particle supported by a massless string (Fig. 15.11). Clearly, the simple pendulum is a better model for the first pendulum illustrated in Fig. 15.10 than for the other two. A simple pendulum can swing back and forth; it can also be given a sideways push and let swing in an orbit around the equilibrium position (Fig. 15.12). Not all real pendulums can carry out such motion, however (refer to Fig. 15.10).

Christian Huygens (1629-1695), the great Dutch contemporary of Newton, made thorough studies of centripetal acceleration while investigating pendulum motion. To Newton's chagrin, Huygens published his studies first in 1673.

Clearly, some energy is stored in the swinging pendulum. As it gradually transfers energy to the air through which it moves and to the support, where there is the inevitable friction, the pendulum's swings become smaller and smaller and finally stop altogether. With some care, it is possible to make a pendulum that swings very many times—50 or 100 times—before coming to rest. The loss of energy during one swing, therefore, is small. It is possible to describe the motion as being approximately periodic (each swing almost repeats the motion during the prior one). The simple pendulum model does not lose energy to other systems, and executes genuine periodic motion. In the following discussion we will first make a mathematical model for the motion of a simple pendulum that executes a circular orbit (Fig. 15.12) and then for one that oscillates through a small angle (Fig. 15.11).

Equation 15.11

force of gravity $= \mathbf{F}_G$
particle mass $= M$
gravitational intensity $= \mathbf{g}$

$$\mathbf{F}_G = M\mathbf{g}$$

Simple pendulum in a circular orbit. Huygens already had studied this system extensively when he was developing pendulum clocks in the seventeenth century. The particle in the simple pendulum is subject to interaction with the earth and with the string. The net force is obtained by adding the two forces. The force of gravity is given in Eq. 15.11. The force exerted by the string is of unknown magnitude, but we do know that it is directed along the string. The net force is a centripetal force directed horizontally. These facts are illustrated in Fig. 15.13 and are sufficient to permit a calculation of the magnitude of the net

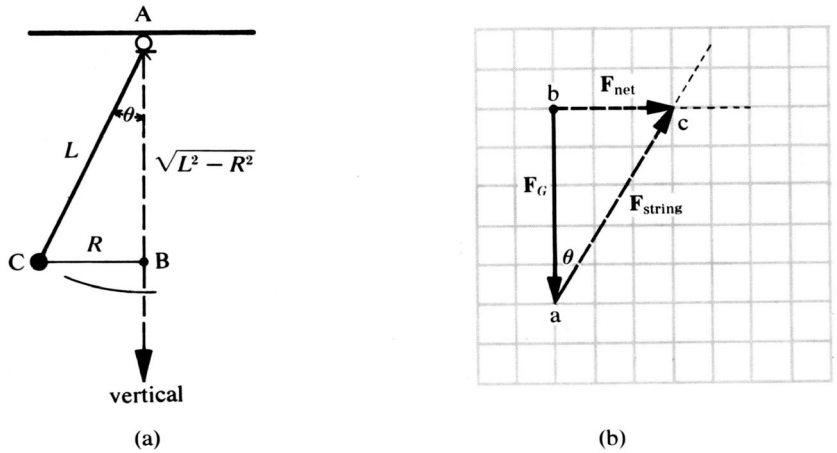


Figure 15.13 Theory of the simple pendulum.
(a) Diagram of the pendulum, showing the string length L, the angle θ, and the radius of the circular orbit R. The triangle ABC is a right triangle with acute angle θ.
(b) Force diagram for the simple pendulum. The dotted lines indicate the directions of the two unknown forces. Their magnitudes are found from the force addition formula, $\mathbf{F}_{(net)} = \mathbf{F}_G + \mathbf{F}_{(string)}$. Triangle abc is similar to triangle ABC.

Equation 15.12

$$\frac{|\mathbf{F}_{net}|}{|\mathbf{F}_G|} = \frac{R}{\sqrt{L^2 - R^2}} \qquad (a)$$

$$|\mathbf{F}_{net}| = \frac{R}{\sqrt{L^2 - R^2}}|\mathbf{g}|M \quad (b)$$

Equation 15.13

$$|\mathbf{F}_{net}| = \frac{R}{\sqrt{L^2 - R^2}}|\mathbf{g}|M$$

$$= \frac{4\pi^2 MR}{\mathcal{T}^2}$$

force by the procedure illustrated in Fig. 11.12 to Fig. 11.14. Since the force triangle (Fig. 15.13b) and the right triangle formed by the string and the vertical (Fig. 15.13a) are similar, the net force magnitude is to the gravitational force magnitude as the radius is to the vertical side (Eq. 15.12). The net force could be measured with a spring scale that is used to hold the pendulum in a deflected position (Fig. 15.14).

The net force given in Eq. 15.12b is then, according to Newton's second law, equal to the centripetal force required for circular motion (Eq. 15.5), resulting in Fig. 15.13. We find, therefore, that the radius of the orbit and the particle mass cancel out, leaving a relations between the length of the pendulum, its period, and the gravitational intensity (Eq.

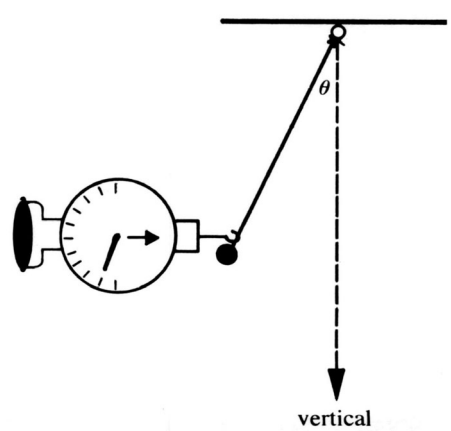


Figure 15.14 Use of spring scale to measure the net force on an orbiting pendulum.

"As to the times of vibration of bodies suspended by threads of different lengths, they bear to each other the same proportion as the square roots of the lengths of the thread; or one might say the lengths are to each other as the squares of the times; so that if one wishes to make the vibration-time of one pendulum twice that of another, he must make its suspension four times as long."

Galileo Galilei
Dialoghi delle Due Nuove Scienze, 1638

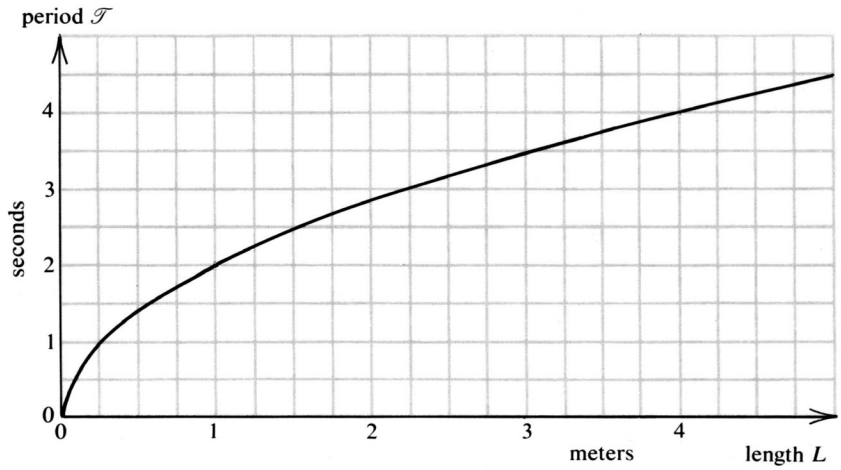


Figure 15.15 Graphical representation of the law of the pendulum at the surface of the earth. The algebraic form of this law is $F = 2.0\sqrt{L}$, where the period T is measured in seconds and the length L in meters.

15.14). For a very small deflection angle (θ), the radius (R) is small compared to the length of the string (L), so that the square root in Eq. 15.14 can be approximated by the length of the string (Eq. 15.15). You can rearrange this formula to express the length in terms of the period (Eq. 15.16a) or the period in terms of the length (Eq. 15.16b). A graph of this relation, called *the law of the pendulum* (discovered by Galileo), is shown in Fig. 15.15. Since the mass canceled out, the relation holds for every simple pendulum at the surface of the earth with a small angle of deflection (θ). Equation 15.16b shows that the period is completely independent of the mass of the particle and, for small angles of swing, the period is also independent of the angle. Thus if a pendulum is started at a small angle, the period will be constant as the pendulum gradually comes to rest. Galileo describes how he first discovered this aspect of the law of the pendulum by using his pulse to time the swings of a hanging church lamp!

Equation 15.14

$$\frac{|\boldsymbol{g}|}{\sqrt{L^2-R^2}}=\frac{4\pi^2}{\mathcal{T}^2}$$

Equation 15.15

$$\sqrt{L^2-R^2}\approx L \qquad \text{(a)}$$

$$\frac{|\boldsymbol{g}|}{L}=\frac{4\pi^2}{\mathcal{T}^2} \qquad \text{(b)}$$

Equation 15.16

$$L\approx\frac{|\boldsymbol{g}|\mathcal{T}^2}{4\pi^2} \qquad \text{(a)}$$

$$\mathcal{T}\approx 2\pi\sqrt{\frac{L}{|\boldsymbol{g}|}} \qquad \text{(b)}$$

The oscillating pendulum. The oscillating motion of a pendulum is observed when the particle is displaced to the side and then released with zero initial speed (Fig. 15.10). The oscillating motion is more difficult to describe mathematically than the circular motion of the orbiting pendulum because the magnitude of the velocity and acceleration are changing all the time. We will therefore not analyze it directly. As long as the angle of deflection (θ) is small, however, the same mathematical models and the same law of the pendulum (Eq. 15.16 and Fig. 15.15) applies to the oscillating pendulum as to the orbiting pendulum.

Applications. Many familiar applications of the simple pendulum make use of the regularity of the orbital or oscillating motion. Both a pendulum clock and a child's swing are characterized by a rhythm that satisfies the law of the pendulum. The period does not change with the width of the swing or the mass of the particle (child). Only

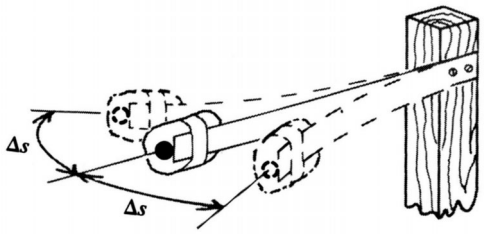


Figure 15.16 One-particle model for an elastic oscillator, The particle of mass M is displaced to a distance Δs from its equilibrium position.

changing the length of the pendulum's support or the gravitational intensity changes the period.

15.5 Elastic oscillators

A weight executes periodic motion when it bounces up and down at the end of a spring. Other examples of periodic motion caused by elastic objects were mentioned in the introduction to this chapter: the inertial balance we used in Section 3.4 to define inertial mass (Fig. 15.16), the bird swaying on a branch of a tree, and the oscillators in an elastic medium, which make wave propagation possible.

We will investigate the motion of a model elastic oscillator in which a massive particle is moving subject to interaction with a "massless" elastic object (a spring, a tree branch) whose behavior is described by Hooke's law (see Section 11.6 and Equation 11.8, in left margin). The one-particle Newtonian approach is adequate for such a working model.

Equation 11.8 (Hooke's Law)

force = **F**
distance displaced = Δs
force constant = κ

$$|\mathbf{F}| = \kappa \Delta s$$

The massive particle has an equilibrium position from which it may be displaced (Fig. 15.16). When it is displaced, the elastic object exerts a restoring force that pulls the particle back to its equilibrium position. For elastic objects described by Hooke's law (Section 11.6), the net force is proportional to the displacement (Eq. 11.8). This situation is similar to the one you encountered with the pendulum (Eq. 15.12b), where the net force was proportional to the radius of the orbit. We therefore select the simple pendulum as an analogue model for the elastic oscillator (Table 15.3).

Equation 15.17 (Elastic oscillator)

Inertial mass (kg) = M
force constant
(newtons/m) = κ
period (sec) = $\mathcal{T}$

$$\mathcal{T} = 2\pi\sqrt{\frac{M}{\kappa}}$$

From the simple pendulum analogue you obtain the mathematical model in Eq. 15.17, according to which the period does not depend on the displacement, but does depend on the mass of the particle and on the strength of the spring. The period increases with greater mass but decreases for stronger springs. This result is consistent with what we found qualitatively in Section 3.4 for the effects of inertia (mass) on the period of the inertial balance.

The most serious limitation of the model is caused by the neglect of the mass of the spring. When the mass of the "massive" particle is equal to zero (no particle is placed on the spring), the model predicts a period of zero seconds. Even an unloaded spring or branch has inertia and therefore an oscillation period unequal to zero, however.

Nevertheless, the model is extremely useful. It can be applied not only to a leaf spring as in the inertial balance, but also to helical springs and coil springs (Fig. 11.23). As a matter of fact, Eq. 15.17 is used to

TABLE 15·3 PENDULUM ANALOGUE FOR ELASTIC OSCILLATOR

Pendulum		*Elastic oscillator*	
particle mass	M	*particle mass*	M
period	$\mathscr{T}$	*period*	$\mathscr{T}$
radius	R	*distance*	Δs
orbital motion		*oscillation*	
net force $\lvert\mathbf{F}\rvert = \frac{\lvert\mathbf{g}\rvert MR}{L}$		*elastic force* $\lvert\mathbf{F}\rvert = \kappa\Delta s$	
law of the pendulum $\mathscr{T} = 2\pi\sqrt{\frac{MR}{\lvert\mathbf{F}\rvert}} = 2\pi\sqrt{\frac{L}{\lvert\mathbf{g}\rvert}}$		*law of the oscillator* $\mathscr{T} = 2\pi\sqrt{\frac{M\Delta s}{\lvert\mathbf{F}\rvert}} = 2\pi\sqrt{\frac{M}{\kappa}}$	

Equation 15.18 (from Eq.15.5) (centripetal force)

centripetal force (newtons) $= |\mathbf{F}|$

radius of circular motion (m) $= R$

period of circular motion (sec) $= \mathscr{T}$

$$|\mathbf{F}| = \frac{4\pi^2 MR}{\mathscr{T}^2}$$

Equation 15.16b
(Law of the pendulum)

$$\mathscr{T} \approx 2\pi\sqrt{\frac{L}{|\mathbf{g}|}}$$

Equation 15.17
(elastic oscillator)

inertial mass (kg) $= M$

strength of interaction (force constant, newtons/m) $= \kappa$

$$\mathscr{T} = 2\pi\sqrt{\frac{M}{\kappa}}$$

determine the force constant of very delicate springs in some spring scales through a measurement of their oscillation periods. Once the force constant is known, the spring scale can be calibrated without resort to the complicated procedure described in Section 11.2. Cavendish used this technique when he measured the force of gravity between lead spheres in his laboratory (Fig. 15.9), and Coulomb used it when he measured the electrical force between charged spheres (Section 11.5).

Summary

The periodic motion of interacting objects played an important part in the history of science. The periodic motion of the planets in the solar system stimulated Newton's development of his theory of moving bodies and the law of gravitation (Eq. 15.10). The mathematical model for centripetal force (Eq. 15.18) was an important intermediate step that enabled Newton to use Kepler's laws of planetary motion in his investigations. The simple pendulum and the elastic oscillator are systems that are used extensively in the regulation of clocks and in scientific studies. The mathematical models governing their motion are given in Eqs. 15.16b and 15.17, respectively.

Additional examples

EXAMPLE 15.3. Calculate the orbital period of a low-altitude satellite (Table 15.2). (Assume that the satellite is sufficiently close to the earth that the radius of its orbit can be assumed to be the same as the radius of the earth.)

Data:

$|\mathbf{g}| = 10$ newtons/kg; $R = 6.6 \times 10^6$ m.

Solution: Near the surface of the earth, the gravitational intensity is 10 newtons per kilogram. This is equal to the satellite's centripetal acceleration.

From Eq. 15·4

$$|\mathbf{g}| = \frac{4\pi^2 R}{\mathscr{T}^2}$$

$$\mathscr{T} = 2\pi\sqrt{\frac{R}{|\mathbf{g}|}} = 2 \times 3.14 \times \sqrt{\frac{6.6 \times 10^6}{10}} \approx 6.3 \times \sqrt{66 \times 10^4}$$

$$= 6.3 \times 8.1 \times 10^2 = 5.1 \times 10^3 \text{ sec} \approx 85 \text{ minutes}$$

EXAMPLE 15·4. Find the mass of the earth.

Data:

$R = 6.4 \times 10^6$ m; $|\mathbf{g}| = 10$ newtons/kg

Solution: Use the law of universal gravitation (Eq. 15·10) and the gravitational intensity of the earth (Eq. 15·9). Call the mass of the earth M_E. The force on a body of mass M at the earth's surface is

$$|\mathbf{F}_G| = |\mathbf{g}|M = \left(G\,\frac{M_E}{R^2}\right) M$$

$$M_E = \frac{R^2|\mathbf{g}|}{G} = \frac{(6.4 \times 10^6)^2 \times 10}{6.7 \times 10^{-11}} \approx 6 \times 10^{24} \text{ kg}$$

EXAMPLE 15·5. What mathematical model for the centripetal force would you infer if Kepler's law had been $\mathscr{T}^2 = KR$?

Solution: Use Eq. 15·18 and insert the hypothesis for $\mathscr{T}^2$

$$|\mathbf{F}| = \frac{4\pi^2 RM}{\mathscr{T}^2} = \frac{4\pi^2 RM}{KR} = \frac{4\pi^2 M}{K}$$

The centripetal force is independent of radius.

EXAMPLE 15·6. What law relating period and radius of circular motion would you predict if the force decreased inversely as the third power of the radius, $|\mathbf{F}| = k/R^3$?

Solution: Use Eq. 15·18 and solve it for the period to the second power, $\mathscr{T}^2$.

$$|\mathbf{F}| = \frac{4\pi^2 MR}{\mathscr{T}^2}, \qquad \mathscr{T}^2 = \frac{4\pi^2 MR}{|\mathbf{F}|}$$

Substitute the force formula

$$\mathscr{T}^2 = \frac{4\pi^2 MR}{k/R^3}$$

$$= \frac{4\pi^2 MR^4}{k}$$

The second power of the period varies as the fourth power of the radius.

EXAMPLE 15·7. A simple pendulum is raised 0.40 meter above its equilibrium position. With what speed will it pass through the equilibrium position?

Data:

$h = 0.40$ m; $|\mathbf{g}| = 10$ m/sec/sec

Solution: Conservation of energy can be used to solve the problem. The mass and length of the pendulum need not be known.

kinetic energy = gravitational energy

$$\tfrac{1}{2} Mv^2 = M|\mathbf{g}|h$$

$$v = \sqrt{2|\mathbf{g}|h} = \sqrt{2 \times 10 \text{ m/sec/sec} \times 0.4 \text{ m}}$$

$$= \sqrt{8.0} = 2.8 \text{ m/sec}$$

EXAMPLE 15·8. In one of Cavendish's experiments (Fig. 15·9), the rod suspended by the thin wire had a vibration period of 15 minutes when the two lead spheres A and A′ at the ends each had a mass of 0.75 kilogram. What is the force constant of Cavendish's thin wire?

Data:

total mass of oscillator $M = 2 \times 0.75$ kg $= 1.5$ kg;
period $\mathscr{T} = 15$ minutes $= 900$ sec

Solution: Use Eq. 15·17.

$$\mathscr{T} = 2\pi\sqrt{\frac{M}{\kappa}}$$

Solving Eq. 15·17 for κ, we get

$$\kappa = 4\pi^2 \frac{M}{\mathscr{T}^2} = 40 \times \frac{1.5 \text{ kg}}{(900 \text{ sec})^2} \approx \frac{60}{8 \times 10^5} = 7.5 \times 10^{-5} \text{ newton/m}$$

(See also Example 15·9.)

EXAMPLE 15·9. What is the deflection of the Cavendish spring scale in Example 15·8 if the large lead spheres B and B′ have a mass of 15 kilograms each and the distance between centers of spheres A and B is 0.13 meter?

Data:

Mass $M_1 = 0.75$ kg; mass $M_2 = 15$ kg; $R = 0.13$ m;
$\kappa = 7.5 \times 10^{-5}$ newton/m; $G = 6.7 \times 10^{-11}$ newton-m²/kg²

Solution: Use Eqs. 11·8 and 15·10.

Note that the gravitational force is that due to *two* pairs of interacting spheres.

$$|\mathbf{F}| = \kappa \Delta s = 2G\,\frac{M_1 M_2}{R^2}$$

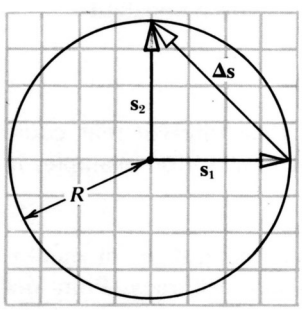
$$= 2 \times 6.7 \times 10^{-11}\,\frac{\text{newton-m}^2}{(\text{kg})^2} \times \frac{0.75\text{ kg} \times 15\text{ kg}}{(0.13\text{ m})^2}$$

$$7.5 \times 10^{-5}\,\Delta s \approx \frac{2 \times 6.7 \times 11 \times 10^{-11}}{1.7 \times 10^{-2}} \approx 8.5 \times 10^{-8}$$

$$\Delta s \approx \frac{8.5 \times 10^{-8}}{7.5 \times 10^{-5}} \approx 1.1 \times 10^{-3}\text{ m} = 1.1\text{ mm}$$

The deflection on the sensitive scale is 1.1 millimeters.

EXAMPLE 15·10. (a) Find the average velocity of a particle in uniform circular motion (radius R) during a time interval $\frac{1}{4}\mathscr{T}$.



Solution: In a time interval $\Delta t = \frac{1}{4}\mathscr{T}$, the particle makes one fourth of a revolution (see diagram in margin). The positions of the particle are

$$\mathbf{s}_1 = [R, 0]$$

$$\mathbf{s}_2 = [0, R]$$

$$\mathbf{\Delta s} = \mathbf{s}_2 - \mathbf{s}_1 = [0, R] - [R, 0] = [-R, R]$$

$$\mathbf{v}_{\text{av}} = \frac{\mathbf{\Delta s}}{\Delta t} = \frac{[-R, R]}{1/4\mathscr{T}} = \left[-\frac{4R}{\mathscr{T}}, \frac{4R}{\mathscr{T}}\right]$$

(b) Find the magnitude of the average velocity in part (a) and compare it with the instantaneous velocity in Eq. 15·2.

Solution:

$$|\mathbf{v}_{\text{av}}| = \sqrt{\left(-\frac{4R}{\mathscr{T}}\right)^2 + \left(\frac{4R}{\mathscr{T}}\right)^2} = \sqrt{\frac{16R^2}{\mathscr{T}^2} + \frac{16R^2}{\mathscr{T}^2}} = \sqrt{\frac{32R^2}{\mathscr{T}^2}}$$

$$= 4\sqrt{2}\,\frac{R}{\mathscr{T}} \approx 5.6\,\frac{R}{\mathscr{T}}$$

From Eq. 15·2,

$$|\mathbf{v}| = \frac{2\pi R}{\mathscr{T}} = 6.28\,\frac{R}{\mathscr{T}}$$

which is greater than

$$|\mathbf{v}_{\text{av}}| = 5.6\,\frac{R}{\mathscr{T}}$$

List of new terms

centripetal acceleration
centripetal force
geocentric
heliocentric
law of universal gravitation
simple pendulum
law of the pendulum
elastic oscillator

List of symbols

a	acceleration	π	3.1415...
$\lvert\mathbf{a}\rvert$	acceleration magnitude	**F**	force
v	velocity	$\lvert\mathbf{F}\rvert$	force magnitude
$\lvert\mathbf{v}\rvert$	velocity magnitude	$\mathbf{F}_G$	force of gravity
v	speed	M	mass
v_{av}	average speed	G	gravitational constant
s	position	**g**	gravitational intensity (acceleration of gravity)
$\Delta\mathbf{s}$	displacement		
Δs	distance		
Δt	time interval	L	length of simple pendulum
$\mathcal{T}$	period		
R	radius of circular orbit	θ	deflection angle
		κ	force constant

Problems

1. Propose two operational definitions of *time interval* that could have been used before the invention of clocks (for example, in ancient times).

2. Undertake library research to determine the history of clocks, especially the gradual improvement of clock accuracy. Point out some of the important uses of clocks at various stages of the development.

3. Calculate the average velocity of a particle moving in a circular orbit of 1 meter radius at a constant speed of 6.28 meters per second. Use the following time intervals: (a) 1 second; (b) 5/6 second; (c) 2/3 second; (d) 1/2 second; (e) 1/3 second; (f) 1/6 second; (g) 1/12 second.

4. (a) Make a graph of the magnitudes of the average velocities in Problem 3 to show their dependence on the time interval.
   (b) Extrapolate to zero time interval on your graph to find the magnitude of the instantaneous velocity.
   (c) Compare the result with that calculated according to Eq. 15.2.

5. Calculate the average acceleration of a particle moving in a circular orbit of 1 meter radius with a constant speed of 6.28 meters per second. Use the following time intervals: (a) 1 second; (b) 5/6 second; (c) 2/3 second; (d) 1/2 second; (e) 1/3 second; (f) 1/6 second; (g) 1/12 second.

6. (a) Make a graph of the magnitudes of the average accelerations vs. the time interval in Problem 5 to show their dependence on the time interval.

(b) Extrapolate to zero time interval on your graph to find the magnitude of the instantaneous acceleration.

(c) Compare the result of the extrapolation with that calculated from Eq. 15.4.

7. The orbital data for the four moons of Jupiter (Io, Europa, Ganymede and Callisto) discovered by Galileo are given below. Determine whether they satisfy Kepler's third law. The units of distance and time are the orbital radius and period of the innermost moon, which are 4 x 10^8 meters and 42.5 hours, respectively

	Io	Europa	Ganymede	Callisto
Orbital radius	1.0	1.6	2.5	4.5
Orbital period	1.0	2.0	4.0	9.5

Optional: Find the mass of Jupiter.

8. (a) Find the centripetal acceleration of a 40-kilogram child on a merry-go-round. The child is at a distance of 5 meters from the center of the merry-go-round, which turns at a rate of six revolutions per minute.
(b) Find the centripetal force on this child.
(c) What object exerts this centripetal force?

9. (a) Obtain or construct an apparatus like that shown in Fig. 15.6 and conduct experiments to test the mathematical model for circular motion (Eq. 15.5).
(b) Discuss some of the sources of experimental error in this experiment.

10. Consult references to determine Copernicus' reasons for preferring a heliocentric over a geocentric model for the solar system. Report and discuss your conclusions.

11. Consult references to find the basis on which Kepler tried to explain planetary motion by forces. Report and discuss his approach.

12. Find the mass of the sun by using data on planetary motion.

13. Explain why the law of gravitation and the moon's orbit around the earth do not allow you to calculate the moon's mass.

14. Verify that the data on the Syncom satellite (Table 15.2) are compatible with the law of gravitation. Use the same method that Newton used, as explained in Section 15.3.

15. Suppose the force of attraction between objects varies inversely as the radius, $|F| = k/R$. What is the form of Kepler's third law appropriate to orbital motion of particles subject to this force?

16. Give four examples from everyday life of systems to which the simple pendulum model should apply.

17. Test one of the systems mentioned in your answer to Problem 16 to determine whether the law of the pendulum applies to it.

18. A large pendulum in a science museum is found to have a period of 8 seconds. How long is the wire supporting it?

19. Find the period of each of the following systems allowed to swing naturally: (a) your right arm held stiffly; (b) your left leg held stiffly; (c) your right forearm (hold the upper arm stationary). Compare and discuss your findings and relate them to your speed of walking. *(Hint:* Measure the time for 20 swings and divide by 20 to find the period.)

20. Give four examples of elastic oscillators from everyday life.

21. Experiment with one of the systems you identified in Problem 20 to find its force constant by: (a) attaching weights or a spring scale and observing the elastic displacement (direct measurement); and (b) making an elastic oscillator and measuring the period (indirect method).

22. Write a critique concerning the application of Eq. 15.17 to two of the systems you selected in answer to Problem 20.

23. (a) Explain under what circumstances you would or would not expect an object hanging from a rubber band to have a period described by the mathematical model in Eq. 15.17.
(b) Test your ideas by experimenting with such an oscillator.

24. Calculate the centripetal acceleration of an object attached to the earth at the equator (due to the rotation of the earth). Compare your result to the acceleration of gravity.

25. Identify one or more explanations or discussions in this chapter that you find inadequate. Describe the general reasons for your judgment (conclusions contradict your ideas, steps in the reasoning have been omitted, words or phrases are meaningless, equations are hard to follow, . . .), and make your criticism as specific as you can.

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